MTH 320: HW 1

Lucky Chouhan

Feburary 5, 2026

Problems from Abbott's book (2nd ed.) are labeled by **Abbott chpt.sec.#.**

1. **(Abbott 1.2.1(b))** Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?

   Solution. Suppose $p/q = \sqrt{4}$ where $p, q \in \mathbb{Z}$ with $q \neq 0$. Then $p = \sqrt{4}q \implies p^2 = 4q^2 \implies p = 2p_1$ then $4p_1^2 = 4q^2 \implies p_1 = \pm q$.

   That is $p = 2p_1 = \pm 2q$ and $p/q = \pm 2$. Here, the contradiction part never happened whereas in the proof of Theorem 1.1.1 we got both p and q even.

2. **(Abbott 1.2.4)** Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

   Solution. Write $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots\}$ and then put every second number starting from 1 in A_1. Now consider $\mathbb{N} \setminus A_1 = \{2, 4, 6, 8, 10, \dots\}$ repeat the same steps as above to get A_2 and then consider $\mathbb{N} \setminus (A_1 \cup A_2)$ to get A_3. Proceeding in this manner, we see that

   $$A_i = \{2^i \cdot (2k-1) : k \in \mathbb{N}\} \quad \text{and} \quad A_i \cap A_j = \emptyset \, \text{for} \, i \neq j.$$

3. **(Abbott 1.2.6 (a)(d))** Prove $||a| - |b|| \leq |a - b|$.

   Solution. Notice that $|a| = |a - b + b| \leq |a - b| + |b|$ and $|b| = |b - a + a| \leq |b - a| + |a|$. These two inequalities give us $|a| - |b| \leq |a - b|$ and $|b| - |a| \leq |b - a| = |a - b|$.

   That is $\max\{|a| - |b|, |b| - |a|\} \leq |a - b|$.

   WLOG assume that $|b| \leq |a|$ then $0 \leq |a| - |b| = \max\{|a| - |b|, |b| - |a|\} \leq |a - b|$. For two positive real numbers x and y with $x \leq y$, we can safely take the absolute value sign both the sides without even touching the "$\leq$" sign. Hence, $||a| - |b|| \leq |a - b|$.

4. **(Abbott 1.2.8)**

   Solution. (a) $f(n) = 2n$ for $n \in \mathbb{N}$.

   (b) $f(x) = \lceil n/2 \rceil$ for $n \in \mathbb{N}$.

   (c) $f(x) = n/2$ for even $n \in \mathbb{N}$ and $f(x) = -(n-1)/2$ for odd $n \in \mathbb{N}$.

5. **(Abbott 1.2.12)** Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

   (a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

   (b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \dots)$ is decreasing.

   Solution. (a) For $n = 1, y_1 = 6 > -6$ is true. Suppose $y_n > -6$ for all natural numbers less than or equal to n. Now, $y_n > -6 \iff 2y_n > -12 \iff 2y_n - 6 > -18 \iff (2y_n - 6)/3 > -6 \iff y_{n+1} > -6$

   Hence, $y_n > -6$ for all $n \in \mathbb{N}$.

   (b) Since $y_1 = 6, y_2 = 2$, so it is true that $y_1 \geq y_2$. Suppose $y_n \geq y_{n+1}$ for all natural numbers less than or equal to n. Consider,

$$\begin{aligned} y_n &\geq y_{n+1} \\ 2y_n &\geq 2y_{n+1} \\ 2y_n - 6 &\geq 2y_{n+1} - 6 \\ (2y_n - 6)/3 &\geq (2y_{n+1} - 6)/3 \\ y_{n+1} &\geq y_{n+2} \end{aligned}$$

   Thus, the sequence (y_n) is a decreasing sequence.

6. Given a function $f : A \to B$ and a subset $C \subset B$. The preimage of C is defined as $f^{-1}(C) = \{a \in A : f(a) \in C\}$. Notice that $f^{-1}(C) \subset A$.

   (a) Let $X \subset A$. Show that $X \subset f^{-1}(f(X))$.

   (b) Let $Y \subset B$. How are $f(f^{-1}(Y))$ and Y generally related?

   Solution. (a) By definition

$$f^{-1}(f(X)) = \{a \in A : f(a) \in f(X)\}.$$

   Let $x \in X$. Then $f(x) \in f(X)$ and thus $x \in f^{-1}(f(X))$.

   (b) By definition

$$f^{-1}(Y) = \{a \in A : f(a) \in Y\}.$$

   Let $y \in f(f^{-1}(Y))$ then there is some $a \in f^{-1}(Y)$ such that $y = f(a)$. This implies that $y = f(a) \in Y$. Therefore, $f(f^{-1}(Y)) \subset Y$.

7. **(Abbott 1.3.3)** (a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbb{R} : b \, \text{is a lower bound for} \, A\}$. Show that $\sup B = \inf A$.

   (b) Use (a) to explain why there is no need to assert that the greatest lower bounds exist as part of the Axiom of Completeness.

   Solution. (a) Since A is nonempty and bounded below, $\inf A$ exist, by the Axiom of Completeness. B is a set of lower bounds of A implies $b \leq a$ for all $a \in A$ and

$b \in B$, means that every element of a is an upper bound for B and by the Axiom of Completeness $\sup B$ exists.

Notice that $\inf A$ is a lower bound for A, so $\inf A \in B$ and any $b \in B$ is lower bound for A but $\inf A$ is greatest lower bound, so $b \leq \inf A$. Thus, $\inf A$ is an upper bound for B.

Combining these two observation: Since $\inf A \in B$ and $\sup B$ is an upper bound for B, we have $\inf A \leq \sup B$. Moreover, $\inf A$ is an upper bound for B and $\sup B$ is the least upper bound, $\sup B \leq \inf A$.

Therefore, $\sup B = \inf A$. $\square$

8. **(Abbott 1.3.4)** Let $A_1, A_2, A_3, \dots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^{n} A_k)$.

(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$ Does the formula in (a) extend to the infinite case?

Solution. It is given that A_k is nonempty and bounded above, $\sup A_k$ exists, by the Axiom of Completeness for all $k \geq 1$.

(a) If $\sup A_1 > \sup A_2$, so there exists some $a \in A_1$ such that $a > \sup A_2$ and $a \in A_1 \cup A_2$ therefore, $\sup(A_1 \cup A_2) = \sup A_1$. Similarly if $\sup A_1 \leq \sup A_2$ then $\sup(A_1 \cup A_2) = \sup A_2$. Thus,

$$\sup(A_1 \cup A_2) = \max\{\sup A_1, \sup A_2\} \tag{1}$$

To extend this to $\sup(\bigcup_{k=1}^{n} A_k)$. Suppose

$$\sup\left(\bigcup_{k=1}^{n} A_k\right) = \max_{1 \leq k \leq n}\{\sup A_k\} \tag{2}$$

for all natural numbers less than or equal to n.

To prove that this is consistent for all natural numbers consider,

$$\begin{aligned}
\sup\left(\bigcup_{k=1}^{n+1} A_k\right) &= \sup\left(\bigcup_{k=1}^{n} A_k \cup A_{n+1}\right) \\
&= \max\left\{\sup\left(\bigcup_{k=1}^{n} A_k\right), \sup A_{n+1}\right\} && \text{(....using (1))} \\
&= \max\left\{\max_{1 \leq k \leq n}\{\sup A_k\}, \sup A_{n+1}\right\} && \text{(....using (2))} \\
&= \max_{1 \leq k \leq n+1}\{\sup A_k\}.
\end{aligned}$$

Thus, $\sup(\bigcup_{k=1}^{n} A_k) = \max_{1 \leq k \leq n}\{\sup A_k\}$ for all $n \in \mathbb{N}$.

(b) For an infinite union, the natural extension would replace "max" with "sup", but **the maximum of an infinite set may not exist**. For example, let $A_k = (0, 1 - 1/k)$.

Thus, we have:

$$\sup\left(\bigcup_{k=1}^{\infty} A_k\right) = \sup\{\sup A_k : k \geq 1\}.$$

Proof. Let $S = \cup_{k=1}^{\infty} A_k$ and $M = \sup\{\sup A_k : k \geq 1\}$.

- For any $x \in S, x \in A_k$ for some k, so $x \leq \sup A_k \leq M$. Hence, M is an upper bound for S, so $\sup S \leq M$.
- For each $k, A_k \subseteq S$, so $\sup A_k \leq \sup S$. Thus, $\sup S$ is an upper bound for $\{\sup A_k\}$, so $M \leq \sup S$.

Therefore, $\sup S = M$.

9. **(Abbott 1.3.6)** Given sets A and B, define $A + B = \{a + b : a \in A \,\text{and}\, b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

   (a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

   (b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

   (c) Finally, show $\sup(A + B) = s + t$.

   (d) Construct another proof of this same fact using Lemma 1.3.8

   Solution. (a) Since $s = \sup A$ and $t = \sup B$ then $a \leq s$ and $b \leq t$ for all $a \in A$ and $b \in B$. Combining the inequalities gives us, $a + b \leq \sup A + \sup B$ for all $a \in A$ and $b \in B$. Notice that $a + b \in A + B$ that means $\sup A + \sup B$ is an upper bound for $A + B$.

   (b) For any fixed $a \in A$, we have $a + B \subseteq A + B$, then,

$$\begin{aligned}\sup(a + B) &\leq \sup(A + B)\\ a + \sup B &\leq \sup(A + B)\\ \sup B &\leq \sup(A + B) - a\end{aligned}$$

   Since u is an arbitrary upper bound for $A + B$ then $\sup(A + B) \leq u$.

   Therefore, $t = \sup B \leq \sup(A + B) - a \leq u - a \implies t \leq u - a$.

   (c) From part (a), $\sup A + \sup B$ is an upper bound for $A + B$, so $\sup(A + B) \leq \sup A + \sup B$. Now let $u = \sup(A + B)$. Then u is an upper bound for $A + B$. By part (b), for any fixed $a \in A$, we have $\sup B \leq u - a$. Rearranging gives $a \leq u - \sup B$. This holds for every $a \in A$, so $u - \sup B$ is an upper bound for A. Hence, $\sup A \leq u - \sup B$, i.e., $\sup A + \sup B \leq u = \sup(A + B)$.

   Combining the two inequalities yield $\sup(A + B) = \sup A + \sup B$.

10. **(Abbott 1.3.8)** Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbb{N}\}$.

(c) $\{n/(3n+1) : n \in \mathbb{N}\}$.

(d) $\{m/(m+n) : m, n \in \mathbb{N}\}$.

Solution. (a) surpema = 1, infima = 0.

(b) surpema = 1, infima = -1.

(c) surpema = 1/3, infima = 1/4.

(d) surpema = 1, infima = 0.