MTH 320: HW 3

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Problems from Abbott's book (2nd ed.) are labeled by **Abbott chpt.sec.#.**

1. State if the following sets are countable or uncountable and give a proof.

   (a) $I = \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers.

   (b) The set of all functions from $\mathbb{N}$ to $\{0, 1\}$.

   Claim (a): $I = \mathbb{R} \setminus \mathbb{Q}$ is uncountable.

   Proof: From $I = \mathbb{R} \setminus \mathbb{Q}$, we see that $\mathbb{R} = \mathbb{Q} \cup I$. We are aware of the fact that the union of any number of countable sets is again a countable set. The uncountability of $\mathbb{R}$ implies that at least one of $\mathbb{Q}$ and I must be uncountable, but we know that $\mathbb{Q}$ is countable. Therefore, I is uncountable.

   Claim (b): The set of all functions from $\mathbb{N}$ to $\{0, 1\}$ is uncountable.

   Proof: A function f from $\mathbb{N}$ to $\{0, 1\}$ can be written as $(a_1, a_2, a_3, \dots)$ where $a_i \in \{0, 1\}$ and $f(n) = a_n$ for all $n \in \mathbb{N}$. We can think of it as a sequence of 0's and 1's.

   Say, S is a set of all such functions. Assume for contradiction that S is countable, then there exists an enumeration of all elements $s_1, s_2, s_3, \dots$ of S.

   Consider the following enumeration,

$$
\begin{aligned}
s_1 &= (a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, \dots) \\
s_2 &= (a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, \dots) \\
s_3 &= (a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, \dots) \\
s_4 &= (a_{41}, a_{42}, a_{43}, a_{44}, a_{45}, \dots) \\
s_5 &= (a_{51}, a_{52}, a_{53}, a_{54}, a_{55}, \dots) \\
&\vdots
\end{aligned}
$$

   Construct a sequence $b = (b_1, b_2, b_3, \dots)$ where,

$$b_n = 1 - a_{nn}.$$

   Since S is countable then there exists $k \in \mathbb{N}$ such that $b = s_k$. Comparing their kth coordinate give us, $b_k = a_{kk}$, but by definition $b_k = 1 - a_{kk}$ and $a_{kk} \neq 1 - a_{kk}$. This is true for all $k \in \mathbb{N}$. Therefore, $b \neq s_k$ for any $k \in \mathbb{N}$. Hence, S is uncountable.

2. Find a function $f : [0, 1) \to (0, 1)$ that is a bijection.

   Solution: Consider sequence $(1/2^n) = (1/2, 1/4, 1/8, \dots)$. We can map $x = 0$ to $1/2$ and $x = 1/2^n$ to $1/2^{n+1}$ and keep every other point fixed.

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \text{ and } x \neq \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{2^{n+1}} & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \end{cases}$$

3. **(Abbott 2.2.2)** Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

   (a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$

   (b) $\lim \frac{\sin(n^2)}{n^{1/3}} = 0$

   Solution (a) Define a sequence (x_n) by $x_n = (2n+1)/(5n+4)$ for all $n \in \mathbb{N}$ and its limit $x = 2/5$. Consider their absolute difference,

$$|x_n - x| = \left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| = \frac{3}{5(5n+4)}$$

   Take $\epsilon > 0$ be any real number and we want $|x_n - x| < \epsilon$ then,

$$\frac{3}{5(5n+4)} < \epsilon \implies n > \frac{1}{5}\left(\frac{3}{5\epsilon} - 4\right).$$

   Thus, for every $\epsilon > 0$ there exists an $N = \lceil \frac{1}{5}\left(\frac{3}{5\epsilon} - 4\right) \rceil$ such that $|x_n - x| < \epsilon$ for all $n \in \mathbb{N}$ with $n > N$.

   Solution (b) Again, define a sequence (x_n) by $x_n = \sin(n^2)/n^{1/3}$ for all $n \in \mathbb{N}$ and notice that $\sin(x) \leq 1$ for all $x \in \mathbb{R}$ implies $x_n \leq 1/n^{1/3}$.

$$|x_n - x| = \left| \frac{\sin(n^2)}{n^{1/3}} - 0 \right| \leq \frac{1}{n^{1/3}}$$

   Take $\epsilon > 0$ be any real number and we want $|x_n - x| < \epsilon$ then,

$$\frac{1}{n^{1/3}} < \epsilon \implies n > \frac{1}{\epsilon^3}.$$

   Thus, for every $\epsilon > 0$ there exists an $N = \lceil \frac{1}{\epsilon^3} \rceil$ such that $|x_n - x| < \epsilon$ for all $n \in \mathbb{N}$ with $n > N$.

4. **(Abbott 2.2.5)**

   (a) Prove that there exists an $N > 0$ large enough such that if $n > N$ then $1 < \frac{12+4n}{3n} < 2$.

(b) Define the sequence $x_n = \lfloor \frac{12+4n}{3n} \rfloor$. Find $\lim_{n\to\infty} x_n$ and give a proof of the convergence.

Proof. (a) Notice that,

$$\frac{12+4n}{3n} = 1 + \frac{n+12}{3n}$$

so it is greater than 1 for all $n \in \mathbb{N}$. Along with this, we want,

$$\frac{12+4n}{3n} < 2 \iff 12+4n < 6n \iff n > 6.$$

For any $n > 6$, the inequality $1 < \frac{12+4n}{3n} < 2$ is true. Hence $N = 6$.

Claim. (b) $\lim_{n\to\infty} = 1$.

Proof. From part (a), we know that $(12+4n)/3n \in (1,2)$ for $n \geq 7$. That means, for all $n \geq 7$, it is of the form $1+\epsilon$ where $\epsilon \in (0,1)$.

Therefore,

$$x_n = \left\lfloor \frac{12+4n}{3n} \right\rfloor = \lfloor 1+\epsilon \rfloor = 1.$$

5. **(Abbott 2.3.1)** Let $x_n \geq 0$ for all $n \in \mathbb{N}$. Show that if $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

   Proof.

   - If $x > 0$: For large n, $\sqrt{x_n}$ is close to $\sqrt{x}$, so eventually $\sqrt{x_n} \geq \sqrt{x}/2$. More rigorously, use the identity

     $$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x}{\sqrt{x}}$$

     Given $\epsilon > 0$, choose N such that $|x_n - x| < \sqrt{x}\epsilon$ for $n \geq N$. Then $\sqrt{x_n} - \sqrt{x}| < \epsilon$.
   - If $x = 0$: Given $\epsilon > 0$, choose N such that $x_n\epsilon^2$ for $n \geq N$ (possible because $x_n \to 0$). Then $\sqrt{x_n} < \epsilon$.

6. **(Abbott 2.3.5)** Let $\{x_n\}$ and $\{y_n\}$ be given. Define the shuffled sequence $\{z_n\}$ by

   $$\{x_1, y_1, x_2, y_2, x_3, y_3, \dots\}.$$

   Prove that the shuffled sequence $\{z_n\}$ is convergent if and only if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

   Proof. Notice that $z_{2k-1} = x_k$ and $z_{2k} = y_k$ for all $k \in \mathbb{N}$.

   Forward direction: Suppose $\lim x_n = \lim y_n = z$.

   Let $\epsilon > 0$, choose $N \in \mathbb{N}$ so that $|x_n - z| < \epsilon$ and $|y_n - z| < \epsilon$ for all $n \geq N$.

   Now take any integer $n \geq M$ with $M = 2N$,

   - if $n = 2k$ (even), and $2k \geq 2N \implies k \geq N$ then $|z_{2k} - z| = |y_k - z| < \epsilon$.

- if $n = 2k-1$ (odd), and $2k-1 \geq 2N \implies k \geq N+1$ then $|z_{2k-1}-z| = |x_k-z| < \epsilon$.

Therefore, $\lim z_n = z$.

Converse direction: Suppose $\lim z_n = z$.

Again, let $\epsilon > 0$, choose $N \in \mathbb{N}$ so that $|z_n - z| < \epsilon$ for all $n \geq N$.

- if $n = 2k - 1$ then, $|z_{2k-1} - z| = |x_k - z| < \epsilon$ gives us $k \geq \lceil (N + 1)/2 \rceil$.
- if $n = 2k$ then, $|z_{2k} - z| = |y_k - z| < \epsilon$ gives us $k \geq \lceil N/2 \rceil$.

Therefore, $|x_k - z| < \epsilon$ and $|y_k - z| < \epsilon$ for all $k \geq \lceil (N + 1)/2 \rceil$.

Thus, $\lim x_n = \lim y_n = z$.