MTH 320: HW 4

Lucky Chouhan

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Problems from Abbott's book (2nd ed.) are labeled by **Abbott chpt.sec.#.**

1. **(Abbott 2.3.6)** Consider the sequence (b_n) given by $b_n = n - \sqrt{n^2+n}$. Prove that (b_n) converges and find its limit. You may use the following facts freely: the Algebraic Limit Laws, $\lim \frac{1}{n} = 0$, and if $\lim x_n = x$ then $\lim \sqrt{x_n} = \sqrt{x}$.

   Solution. Rationalizing the sequence (b_n) gives,

$$b_n = \frac{-n}{n+\sqrt{n^2+n}} = \frac{-1}{1+\sqrt{1+1/n}}$$

   Notice that for any $n \in \mathbb{N}, 1 + 1/n \geq 1 + 1/(n+1) \implies b_n \geq b_{n+1}$. Further,

$$\begin{aligned} 1+\frac{1}{n} &\geq 1 \\ 1+\sqrt{1+\frac{1}{n}} &\geq 2 \\ b_n &\geq -\frac{1}{2}. \end{aligned}$$

   Thus, (b_n) is decreasing and bounded. Therefore, by Monotone Convergence Theorem it converges.

   Using algebraic properties of limits of sequences we see that,

$$\begin{aligned} \lim(b_n) &= \lim\left(\frac{-1}{1+\sqrt{1+1/n}}\right) \\ &= \frac{\lim(-1)}{\lim(1+\sqrt{1+1/n})} \\ \lim(b_n) &= -\frac{1}{1+1} = -\frac{1}{2}. \end{aligned}$$

2. **(Abbott 2.3.12)** In the following assume that (a_n) converges to a and determine the validity of each claim. If it is true then provide a proof and if it is false provide a counterexample.

(a) Let $X \subset \mathbb{R}$. If a_n is an upper bound for X then a is an upper bound for X.

(b) If a_n is in the complement of the set $(0,1)$ for all n then a is in the complement of $(0,1)$.

(c) If a_n is rational for all n then a is rational.

Claim (a). The claim is true.

Proof. For any $x \in X$, we have $a_n \geq x$. Define $b_n = a_n - x \geq 0$. From algebraic properties of limit $\lim(b_n) = \lim(a_n - x) = \lim(a_n) - \lim(x) = a - x$.

Take $b = a - x$ and our goal is to prove that $b \geq 0$. Assume that $b < 0$. Since (b_n) converges to b then for $\epsilon = |b|/2$ there exists an $N \in \mathbb{N}$ so that $|b_n - b| < |b|/2$ for all $n \geq N$ that is $b - |b|/2 < b_n < b - |b|/2 < 0$.

But this contradicts the fact that $b_n \geq 0$ for all n. Therefore, $a - x = b \geq 0$ and $a \geq x$.

Claim (b). The claim is true.

Proof. Suppose for contradiction, the $a \in (0,1)$, Let $\epsilon = \min\{a, 1-a\}/2 > 0$. Then the ϵ-neighborhood $(a-\epsilon, a+\epsilon)$ is contained in $(0,1)$. Since $a_n \to a$, there exists N such that for all $n \geq N$, $|a_n - a| < \epsilon$, hence $a_n \in (0,1)$. This contradicts the hypothesis that no a_n lies in $(0,1)$. Therefore $a \in (0,1)$.

Claim (c). The claim is false. **Counterexamples.** We can define sequence of rationals $x_n = \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N}$ and we also know that $\lim x_n = e$ that is an irrational number.

Or we could have defined $y_n = \lfloor 10^n \sqrt{2} \rfloor 10^{-n}$ for $n \in \mathbb{N}$ and we can see that $y_n \in \mathbb{Q}$ for each n but $\lim y_n = \sqrt{2} \notin \mathbf{Q}$.

3. **(Abbott 2.4.1)**

(a) Let (x_n) be a sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}.$$

Prove that (x_n) converges.

(b) From (a) we know $\lim x_n$ exists. Why must x_{n+1} also exists and be equal to the same value?

(c) Take the limit on both sides of the recursive equation in part (a) and explicitly compute $\lim x_n$.

Proof. (a) Assume that $x_n \leq 1$ for all $n \geq 2$, evaluating $x_2 = 1, x_3 = 1/3$ agree with our assumption, $x_n \leq 1 \implies 4 - x_n \geq 3 \implies x_{n+1} \leq 1/3 \leq 1$.

Now also assume that $x_n \geq 0$ for all n. For $n+1$, $4 - x_n \leq 4 \implies x_{n+1} \geq 1/4 \geq 0$.

Therefore, $x_n \in [1/4, 1]$ for all $n \geq 2$.

Consider the difference of two consecutive terms of sequence,

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{4 - x_n} - \frac{1}{4 - x_{n-1}} \\ &= \frac{x_n - x_{n-1}}{(4 - x_n)(4 - x_{n-1})} \end{aligned}$$

recursively applying the last relation gives us,

$$\begin{aligned} x_{n+1} - x_n &= \frac{-2}{(4 - x_n)(4 - x_{n-1})^2(4 - x_{n-2})^2 \cdots (4 - x_2)^2} \\ &= -\underbrace{\left(\frac{2 \cdot x_{n+1}}{(4 - x_{n-1})^2(4 - x_{n-2})^2 \cdots (4 - x_2)^2}\right)}_{\geq 0} \\ &\leq 0. \end{aligned}$$

Thus, (x_n) is a decreasing sequence bounded above by 1, therefore by Monotone Convergence Theorem it converges.

Explanation (b). Since we know (x_n) converges, say to x, that means, for any $\epsilon > 0$, we can find an $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq N$. Moreover, $n + 1 \geq n \geq N$, therefore, $|x_{n+1} - x| < \epsilon$ for each $\epsilon > 0$. Hence, $\lim(x_{n+1}) = \lim(x_n) = x$.

Computation (c). Take $\lim x_n = x$. We've,

$$x_{n+1} = \frac{1}{4 - x_n}.$$

So,

$$x = \lim(x_{n+1}) = \lim\left(\frac{1}{4 - x_n}\right) = \frac{1}{4 - x}.$$

Solving, the equation $x^2 - 4x + 1 = 0$ gives us $x = 2 \pm \sqrt{3}$. Since $2 + \sqrt{3} = 3.732 \cdots \geq 3 \geq x_n$ for all n, therefore, $x = 2 - \sqrt{3} = 0.2679 \ldots$.

4. **(Abbott 2.4.3)** Prove that the sequence (x_n) defined by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ converges.

   Proof. Notice that, $x_1 = \sqrt{2} = 2\cos(\pi/4)$ and using $1 + \cos(\theta) = 2\cos^2(\theta/2)$ for any θ, we see that,

   $$x_2 = \sqrt{2 + x_1} = \sqrt{2 + 2\cos(\pi/4)} = \sqrt{4\cos^2(\pi/8)} = 2\cos(\pi/8).$$

   Iterating this for all, we see that $x_n = 2\cos(\frac{\pi}{2^{n+1}})$ for any n. For n + 1,

   $$x_{n+1} = \sqrt{2 + x_n} = \sqrt{2 + 2\cos\left(\frac{\pi}{2^{n+1}}\right)} = 2\cos\left(\frac{\pi}{2^{n+2}}\right).$$

Further, $\frac{\pi}{2^{n+1}} > \frac{\pi}{2^{n+2}}$ and we know that $\cos(x)$ is a decreasing function in $[0, \pi]$ therefore, $\cos\left(\frac{\pi}{2^{n+1}}\right) \leq \cos\left(\frac{\pi}{2^{n+2}}\right)$ that is $x_n \leq x_{n+1}$.

Notice that, $\cos(x) \leq 1$, so is x_n. Thus, by Monotone Convergence Theorem x_n converges and its limit can be evaluated as,

$$\lim x_n = \lim \left(2\cos\left(\frac{\pi}{2^{n+1}}\right)\right) = 2.$$

5. **(Abbott 2.4.7)** Let (a_n) be a bounded sequence. Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges. (**Remark:** The limit superior of (a_n) is defined by $\limsup a_n = \lim y_n$.)

   Proof. Since (a_n) is bounded, so there exists $M \in \mathbb{N}$ such that $a_n \geq M$ for all $n \in \mathbb{N}$. Moreover, $y_n = \sup\{a_k : k \geq n\} \geq a_n \geq M$, this tells us (y_n) is also bounded below by M.

   We can also see y_n as,

$$\begin{aligned} y_n &= \sup\{a_k : k \geq n\} \\ &= \max\{a_n, \sup\{a_k : k \geq n+1\}\} \\ &= \max\{a_n, y_{n+1}\} \\ &\geq y_{n+1}. \end{aligned}$$

   Thus, (y_n) is decreasing and bounded sequence, therefore by Monotone Convergence Theorem converges.

6. Establish the convergence of the sequence (y_n) defined by

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}.$$

   Proof. Notice that,

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \leq \frac{n}{n+1} \leq 1.$$

   So, the sequence (y_n) is bounded above by 1. Further,

$$\begin{aligned} y_{n+1} - y_n &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{1}{2n+1} - \frac{1}{2(n+1)} \\ &= \frac{1}{2(n+1)(2n+1)} \\ y_{n+1} - y_n &\geq 0. \end{aligned}$$

   From the above inequality, we see that (y_n) is an increasing sequence. Therefore, by Monotone Convergence Theorem, it converges.

7. **(Abbott 2.4.2)** Consider the sequence defined by $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Prove that (y_n) converges and find $y = \lim y_n$.

   Proof. Computing few terms of (y_n) suggests us that it is an increasing sequence bounded above by 3.

   Suppose $y_n \geq y_{n-1}$ for some n then $y_{n+1} = 3 - \frac{1}{y_n} \geq 3 - \frac{1}{y_{n-1}} = y_n$. Thus, $y_{n+1} \geq y_n$ for all $n \in \mathbb{N}$.

   Suppose $y_n \leq 3$, and (y_n) is an increasing sequence with $y_1 = 1$ so $1/y_n \geq 1/3$ then $y_{n+1} = 3 - \frac{1}{y_n} \leq 3 - \frac{1}{3} < 3$.

   Therefore, by Monotone Convergence Theorem it converges. Taking the limit the the side of $y_{n+1} = 3 - \frac{1}{y_n}$ gives us, $y^2 - 3y + 1 = 0$, and its solutions are $y = 2.618033\ldots$ and $y = 0.381966\ldots$. Rejecting $y = 0.381966\ldots$ gives us $y = \lim y_n = 2.618033\ldots$.