MTH 320: HW 3

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Due: January 30, 2019

Problems from Abbott's book (2nd ed.) are labeled by **Abbott chpt.sec.#.**

1. State if the following sets are countable or uncountable and give a proof.
   (a) $I = \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers.
   (b) The set of all functions from $\mathbb{N}$ to $\{0, 1\}$.
2. Find a function $f : [0, 1) \to (0, 1)$ that is a bijection.
   (**Hint.** Define a function by
   $$f(x) = \begin{cases} x & \text{if } x \neq \frac{1}{n} \\ \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \end{cases}.$$
   Show that f is a bijection. Note that this shows that $[0, 1) \sim (0, 1) \sim \mathbb{R}$. With a little more effort one can find a bijection from $[0, 1]$ to $\mathbb{R}$.)
3. **(Abbott 2.2.2)** Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.
   (a) $\lim \dfrac{2n+1}{5n+4} = \dfrac{2}{5}$
   (b) $\lim \dfrac{\sin(n^2)}{n^{1/3}} = 0$ (You may use Theorem 2.3.4 which I will cover on Monday).
4. **(Abbott 2.2.5)**
   (a) Prove that there exists an $N > 0$ large enough such that if $n > N$ then $1 < \frac{12+4n}{3n} < 2$.
   (b) Define the sequence $a_n = \left\lfloor \frac{12+4n}{3n} \right\rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Find $\lim_{n\to\infty} a_n$ and give a proof of the convergence.
5. **(Abbott 2.3.1)** Let $x_n \geq 0$ for all $n \in \mathbb{N}$. Show that if $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$. (**Hint:** Use that $(\sqrt{x_n} - \sqrt{x})(\sqrt{x} + \sqrt{x_n}) = x_n - x$.)
6. **(Abbott 2.3.5)** Let $\{x_n\}$ and $\{y_n\}$ be given. Define the shuffled sequence $\{z_n\}$ by
   $$\{x_1, y_1, x_2, y_2, x_3, y_3, \ldots\}.$$
   Prove that the shuffled sequence $\{z_n\}$ is convergent if and only if $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$.