MTH 320: HW 4

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Due: February 6, 2019

Problems from Abbott's book (2nd ed.) are labeled by **Abbott chpt.sec.#.**

1. **(Abbott 2.3.6)** Consider the sequence (b_n) given by $b_n = n - \sqrt{n^2 + n}$. Prove that (b_n) converges and find its limit. You may use the following facts freely: the Algebraic Limit Laws, $\lim \frac{1}{n} = 0$, and if $\lim x_n = x$ then $\lim \sqrt{x_n} = \sqrt{x}$.

2. **(Abbott 2.3.12)** In the following assume that (a_n) converges to a and determine the validity of each claim. If it is true then provide a proof and if it is false provide a counterexample.

   (a) Let $X \subset \mathbb{R}$. If a_n is an upper bound for X then a is an upperbound for X.

   (b) If a_n is in the complement of the set $(0, 1)$ for all n then a is in the complement of $(0, 1)$.

   (c) If a_n is rational for all n then a is rational.

3. **(Abbott 2.4.1)**

   (a) Let (x_n) be a sequence defined by $x_1 = 3$ and
   $$x_{n+1} = \frac{1}{4 - x_n}.$$
   Prove that (x_n) converges.

   (b) From (a) we know $\lim x_n$ exists. Why must $\lim x_{n+1}$ also exists and be equal to the same value?

   (c) Take the limit on both sides of the recursive equation in part (a) and explicitly compute $\lim x_n$.

4. **(Abbott 2.4.3)** Prove that the sequence (x_n) defined by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ converges.

5. **(Abbott 2.4.7)** Let (a_n) be a bounded sequence. Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges. (**Remark:** The limit superior of (a_n) is defined by $\limsup a_n = \lim y_n$.)

6. Establish the convergence of the sequence (y_n) defined by
   $$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n}.$$

Suggested problem (not to be collected)

(Abbott 2.4.2) Consider the sequence defined by $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Prove that (y_n) converges and find $y = \lim y_n$.
Solution (sketch).

- Use induction to show that $y_n < 3$, that is the sequence is bounded above.
- Show that the sequence is monotone increasing, $y_n \leq y_{n+1}$.
- Apply monotone convergence theorem.
- Take limit on both sides of the recurrence relation and solve it for the limit.