MTH 320: HW 5

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Due: February 13, 2019

Problems from Abbott's book (2nd ed.) are labeled by **Abbott chpt.sec.#.**

1. **(Abbott 2.5.2)** Decide whether the following propositions are true or false, providing a short justification for each conclusion (full proof is not needed but please provide a convincing argument).

   (a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

   (b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

   (c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

   (d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

2. **(Abbott 2.5.5)** Assume that (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

3. Let (a_n) be a bounded sequence. Recall from HW4, we defined a sequence $y_n = \sup\{a_k : k \geq n\}$ and proved that (y_n) is bounded and monotone. Thus, by the Monotone Convergence Theorem (y_n) converges. Let $y = \lim_{n\to\infty} y_n$.

   Prove that there exists a subsequence (a_{n_k}) of (a_n) that converges to y.

4. **(Abbott 2.6.3)** If (x_n) and (y_n) are Cauchy sequences prove directly that $(x_n y_n)$ is a Cauchy sequence.

5. (a) Show directly from the definition that $\left(\dfrac{n+1}{n}\right)$ is a Cauchy sequence.

   (b) Show directly from the definition that $\left(n + \dfrac{(-1)^n}{n}\right)$ is not a Cauchy sequence.

6. Suppose that (x_n) is a sequence such that $|x_{n+1} - x_n| < \dfrac{1}{n^2}$ for all $n \in \mathbb{N}$. Prove that (x_n) is a Cauchy sequence.

7. **(Abbott 2.7.1)** Suppose that (a_n) is a decreasing sequence and that $\lim_{n\to\infty} a_n = 0$. Let

$$s_N = a_1 - a_2 + a_3 - \ldots + (-1)^{N+1} a_N = \sum_{n=1}^{N} (-1)^{n+1} a_n.$$

   Prove directly that (s_N) is a Cauchy sequence.

8. **(Abbott 2.7.8)** Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

   (a) If $\sum a_n$ converges absolutely, then also $\sum a_n^2$ converges absolutely.

   (b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

   (c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.