Rearrangements of Infinite Series

This handout presents the fundamental results on convergence/divergence of rearrangements of infinite series. It shows that a conditionally convergent series can be rearranged so as to diverge, and it gives a proof, different from the one in our text, that every rearrangement of an absolutely convergent series converges to the same sum.

1 Permutations of sets

A permutation of a set X is a function $f : X \to X$ that is one-to-one and takes X onto itself.

1.1 Example. $X = \{1, 2, 3\}$, $f(1) = 2$, $f(2) = 3$, $f(3) = 1$ (a "cyclic permutation").

We'll express this (and all following) examples more succinctly like so:

$$f : (1, 2, 3) \to (2, 3, 1)$$

or, omitting explicit reference to f, just:

$$(1, 2, 3) \to (2, 3, 1).$$

More generally you can construct permutations of $\mathbb{N}$ by permuting the first few positive integers and leaving the rest alone, e.g.:

$$(1, 2, 3, 4, 5, 6, \ \ldots) \to (2, 3, 1, 4, 5, 6, \ \ldots).$$

Here is a more interesting permutation of $\mathbb{N}$:

1.2 Example. $(1, 2, 3, 4, 5, 6, 7, 8, \ \ldots) \to (2, 1, 4, 6, 3, 8, 10, 12, 5, \ldots)$.

Here we write down the first even (positive) integer, then the first odd one, then the next two evens, then the next odd, then the next three evens, then the next odd, etc.

1.3 Rearrangement of sequences and series A *rearrangement* of a sequence $(a_n)_1^\infty$ is just a permutation of the terms of the sequence. More precisely, to say that (b_n) *is a rearrangement of* (a_n) means that there is a permutation f of $\mathbb{N}$ for which

$$b_n = a_{f(n)} \qquad \forall \ n \in \mathbb{N}.$$

When this happens we also say that the infinite series $\sum_{n=1}^\infty b_n$ is a *rearrangement* of the series $\sum_{n=1}^\infty a_n$.

2 The Series Rearrangement Theorems

This next section is devoted to the two most important results on convergence of rearrangements of series.

2.1 Theorem (SRT 1). *Every conditionally convergent series has a rearrangement that diverges.*

2.2 Theorem (SRT 2). *Every rearrangement of an absolutely convergent series converges (absolutely) to the same sum.*

The proofs of these results depend crucially on Exercise 2.7.3 on page 68 of our text. Here's a restatement of that exercise—in a slightly different form (our q_n's will be the negatives of the book's).

2.3 Lemma. For a sequence $(a_n)_1^\infty$ of real numbers, define, for each $n \in \mathbb{N}$,

$$p_n = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases} \qquad \text{and} \qquad q_n = \begin{cases} -a_n & \text{if } a_n < 0 \\ 0 & \text{if } a_n > 0. \end{cases}$$

Thus, for each $n \in \mathbb{N}$, both p_n and q_n are ≥ 0, and I leave it to you to check that, in addition:

$$p_n + q_n = |a_n| \qquad (n \in \mathbb{N}), \tag{1}$$

and

$$p_n - q_n = a_n \qquad (n \in \mathbb{N}). \tag{2}$$

Note that from (1) and the "Series Algebra Theorem" (Theorem 2.7.1) we immediately conclude the following:

$$\sum_n |a_n| \text{ converges } \Leftrightarrow \sum_n p_n \text{ and } \sum_n q_n \text{ converge.} \tag{3}$$

A little more subtle is:

$$\sum_n a_n \text{ converges conditionally } \Leftrightarrow \text{it converges but } \sum_n p_n \text{ and } \sum_n q_n \text{ both diverge.} \tag{4}$$

To prove this, note that by (2), $q_n = p_n - a_n$, so if $\sum_n p_n$ and $\sum_n a_n$ both converge, then by the Series Algebra Theorem, so must $\sum_n q_n$. Thus if $\sum_n a_n$ converges and $\sum_n q_n$ diverges, then $\sum_n p_n$ must diverge also. The same is true with q_n in place of p_n, which proves (4). $\square$

2.4 Proof of SRT 1. Suppose $\sum_n a_n$ converges conditionally. Then both its "positive-weight series" $\sum_n p_n$ and its "negative-weight series" $\sum_n q_n$ diverge, by (4) above. We'll concentrate on the positive-weight series.

Start by adding in consecutive positive weights p_n until you get a sum > 1. Then add in $-q_1$, reducing the sum a bit. Then build up the sum with consecutive remaining positive weights until it's > 2. then add in $-q_2$. In this way we create a rearrangement of the original series $\sum a_n$, a subsequence of whose partial sums is unbounded (in fact, since $q_n \to 0$, the whole sequence of partial sums is unbounded, but that's not important right now). Thus the the rearrangement diverges. $\square$

2.5 Proof of SRT 2. We're now assuming that $\sum_n |a_n|$ converges, hence so does $\sum_n a_n$. We want to show that every rearrangement of this latter series converges, and to the same sum as the original.

SPECIAL CASE: *Suppose* $a_n \geq 0$ *for all* $n \in \mathbb{N}$. Then the partial sums (s_n) of $\sum_n a_n$ form an increasing, bounded sequence whose limit s (the sum of the series $\sum_n a_n$) is also the supremum of the set $\{s_n : n \in \mathbb{N}\}$. Let f be a permutation of $\mathbb{N}$ and let (t_n) be the sequence of partial sums of the rearranged series $\sum_n a_{f(n)}$.

Then (t_n) is also an increasing sequence.

Claim: $t_n \leq s$ for each n.

Indeed, given $n \in \mathbb{N}$ we can find $M(n) \in \mathbb{N}$ for which

$$\{f(1), f(2), \ \ldots \ f(n)\} \subset \{1, 2, \ \ldots \ M(n)\}$$

(just take $M(n) = \max\{f(1), f(2), \ \ldots \ f(n)\}$). Thus

$$t_n \leq s_{M(n)} \leq s,$$

as desired.

This shows that the rearranged series converges, and its sum—call it t—is $\leq s$.

SUMMARIZING: *If a positive-term series converges, then any rearrangement converges to a sum* $\leq$ *the original sum.*

But $\sum_n a_n$ is a rearrangement of $\sum_n a_{f(n)}$, hence its sum is $\leq t$. Thus $s = t$ and the proof is finished for positive-term series.

THE GENERAL CASE: We no longer suppose each a_n to be ≥ 0. By (3) we know that both the positive-weight series $\sum_n p_n$ and the negative-weight series $\sum_n q_n$ converge, and, by the Series Algebra Theorem, that that $\sum_n a_n = \sum_n p_n - \sum_n q_n$. Any rearrangement of $\sum_n a_n$ induces rearrangements of the positve- and negative-weight series', and by the "special case" above, these rearrangements converge to the sums of the unrearranged series. Thus the rearranged series, which is the sum of the rearranged positve- and negative-weight series', also converges to the original sum. $\square$

3 Exercises

1. Modify the argument used to prove SRT 1 to show that any conditionally convergent series can be rearranged so as to converge to any number you wish.

2. Modify the argument used to prove SRT 1 to show that if A and B are real numbers, then any conditionally convergent series can be rearranged so that a subsequence of its partial sums converges to A and another subsequence to B.

3. For $f : \mathbb{N} \to \mathbb{N}$ as in §1.2, show that the series $\sum_{n=0}^{\infty} 2^{-f(n)}$ converges, and find its sum.

4. Use the "positive-weight, negative-weight" idea to give another proof that every absolutely convergent series is convergent.

5. *Inserting or removing parentheses.* (a) Show that if a series converges then the new series you get by "inserting parentheses" in the original one (i.e., adding up finite blocks of consecutive terms) converges to the same sum.

   (b) Give a (simple) example to show that *removing* parentheses can change a convergent sequence into a divergent one.

6. In the author's proof that every rearrangement of an absolutely convergent series converges to the same sum, there's an error in the formula for M (1/3 the way down page 67). What is it, and how do you correct it?