The Alternating Series Test

This handout presents the solution to Exercise #2.7.1, which asks for a proof of the Alternating Series Test using the Cauchy Criterion for series (Theorem 2.7.2).

The infinite series in question is:

$$a_1 - a_2 + a_3 - a_4 + \quad \cdots \quad \text{i.e.,} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n \quad . \tag{1}$$

Theorem 2.7.7 (The Alternating Series Test). *If $a_n \searrow 0$ then the series (1) converges.*

Proof. STEP I. Let s_n denote the n-th partial sum of the series, i.e.

$$s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k \;=\; a_1 - a_2 + a_3 - a_4 + \quad \cdots \quad \pm a_n \ ,$$

where in the last term on the right-hand side we choose the plus sign if n is odd and the minus sign if n is even.

Note that $s_1 = a_1 \geq 0$, and that $s_2 = a_1 - a_2$ is also ≥ 0 (since $a_2 \leq a_1$), and $s_2 \leq s_1$ (since $a_2 \geq 0$). So far,

$$0 \leq s_2 \leq s_1 = a_1 \ . \tag{2}$$

Let's go one more step for practice: $s_3 = a_1 - a_2 + a_3 = s_2 + a_3$, so $s_2 \leq s_3$ (because $a_3 \geq 0$). But also $s_3 = s_1 - (a_2 - a_3) \leq s_1$, because $a_2 - a_3 \geq 0$. Thus

$$0 \leq s_2 \leq s_3 \leq s_1.$$

Now I leave it to you to prove (by five separate little induction arguments) that:

(i) All the partial sums s_n of the series (1) are non-negative.

(ii) All the partial sums are $\leq$ the first term, a_1.

(iii) The even-indexed partial sums form an increasing sequence.

(iv) The odd-indexed partial sums form a decreasing sequence.

(v) Each even-indexed partial sum is $\leq$ all the odd-indexed partial sums.

STEP II. Now to prove our Theorem: Let $\varepsilon > 0$ be given. By the Cauchy Criterion for Series (Theorem 2.7.2), our goal is to find $N \in \mathbb{N}$ so that whenever $n, m \in \mathbb{N}$ with $N \leq m < n$ then

$$|s_n - s_m| = |a_{m+1} - a_{m+2} + a_{m+3} - \quad \cdots \quad \pm a_n| < \varepsilon. \tag{3}$$

By (i) above, the absolute values on the right-hand side of this equation can be removed, and by (ii), now applied to the alternating series starting, not with a_1, but with a_{m+1}, this right-hand side is $\leq a_{m+1}$. Now because $a_n \searrow 0$ we can find $N \in \mathbb{N}$ such that $a_n < \varepsilon$ whenever $n \geq N$. Thus

$$N \leq m < n \Rightarrow |s_n - s_m| \leq a_m < \varepsilon,$$

hence (3) is satisfied, so the sequence of partial sums (s_n) of our series (1) is Cauchy, and therefore convergent. Thus the series itself is convergent. □

Here are a couple of important corollaries not mentioned in the text:

Corollary 1. *If $a_n \searrow 0$ then the alternating series (1) converges to a non-negative sum that is $\leq a_1$.*

Proof. By (i) and (ii) of STEP I above, each partial sum lies between 0 and a_1. Thus the same is true of the limit of these partial sums (by the "Order Limit Theorem," Theorem 2.3.4). □

Corollary 2. *If $a_n \searrow 0$, and s denotes the sum of the alternating series (1), then*

$$|s - s_n| \leq a_{n+1} \qquad (\forall\, n \in \mathbb{N}).$$

Proof. I leave it to you to check that $|s - s_n|$ is given by the convergent series

$$a_{n+1} - a_{n+2} + a_{n+3} + \ \ldots\ ,$$

which is itself an alternating series of the form (1). The previous Corollary says that for any such series, the sum lies between 0 and the first term, which, in this case is a_{n+1}. □.

Example. *Find a partial sum that approximates the actual sum of the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \cdots$ with accuracy at least 10^{-6}.*

Solution. By Corollary 2 above the 999,999-th partial sum does the job! □