Sequences and Series

2.1. The Limit of a Sequence

Definition 2.1. A **sequence** is a function whose domain is $\mathbf{N}$. If this function is denoted by f, then the values $f(n)$ $(n \in \mathbf{N})$ determine the sequence uniquely, and vise-versa. Therefore, a sequence can also be denoted by

$$(a_1, a_2, a_3, a_4, \cdots) \quad \text{or} \quad (a_n)_{n=1}^{\infty},$$

where $a_n = f(n)$ for $n \in \mathbf{N}$.

Throughout this course we only study sequences of real numbers; namely functions $f: \mathbf{N} \to \mathbf{R}$.

Example 2.1. Each of the following are common ways to describe a sequence.

(i) $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$,

(ii) $(\frac{n}{n+1})_{n=1}^{\infty}$,

(iii) (a_n), where $a_n = 2^n$ for all $n \in \mathbf{N}$,

(iv) (x_n), where $x_1 = 2$ and $x_{n+1} = \frac{x_n+1}{2}$. This is the *induction* or *recursion* way to define a sequence.

Example 2.2. Notice the difference between a sequence (a_n) and a set $\{a_n : n \in \mathbf{N}\}$:

$((-1)^n)_{n=1}^{\infty} = (-1, 1, -1, 1, -1, 1, \cdots)$ is a sequence, having infinitely many terms (which can have repeated values);

$\{(-1)^n : n \in \mathbf{N}\} = \{1, -1\}$ is simply a set of two elements, not a countable set nor a sequence;

$(c) = (c, c, c, c, \cdots)$ is the constant sequence; $\{c\}$ is the set of single element c.

Definition 2.2 (Convergence of Sequences)**.** A sequence (a_n) is said to **converge** to a real number a (called the **limit** of the sequence) if, for every number $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n \in \mathbf{N}$ and $n \geq N$ it follows that $|a_n - a| < \epsilon$. That is,

$$\forall \epsilon > 0 \; \exists N \in \mathbf{N} \; \forall n \in \mathbf{N} \; (n \geq N \Longrightarrow |a_n - a| < \epsilon).$$

In this case we write either $\lim a_n = a$ or $a_n \to a$.

If a sequence (a_n) does not converge to any real number, then we say that the sequence (a_n) **diverges**.

Easy Fact: $\lim(c) = c$ for all constant sequences (c).

Quantifiers. The definition of $\lim a_n = a$ quantifies the closeness of a_n to a by an arbitrarily given $\epsilon > 0$ (*universal quantifier*) and the truth of this closeness for all terms after a term a_N (N is the *universal quantifier*). It is often the case the smaller ϵ is the larger N is needed to be. However the heart of the matter in the definition is that: no matter how small $\epsilon > 0$ will there always exist such an integer N validating the requirement: $\forall\, n \geq N$ ($|a_n - a| < \epsilon$).

Template for a proof of $\lim a_n = a$

- Let $\epsilon > 0$ be arbitrary (not 1 or $\frac{1}{1000^5}$, but arbitrary; no numerical values are known).
- Try to solve the inequality $|a_n - a| < \epsilon$ to determine how to choose an $N \in \mathbf{N}$ so that this inequality holds for all $n \geq N$. This step usually requires the most work, almost of all of which is done prior to actually writing the formal proof.
- Now show that the N found actually works; namely for all integers $n \geq N$ the inequality $|a_n - a| < \epsilon$ indeed holds.

Example 2.3. Show

$$\lim(\frac{n+1}{n}) = 1.$$

Proof. Let $a_n = \frac{n+1}{n}$ and $a = 1$. Then the inequality

$$|a_n - a| = \left|\frac{n+1}{n} - 1\right| = \frac{1}{n} < \epsilon$$

is the same as $n > \frac{1}{\epsilon}$. The existence of $N \in \mathbf{N}$ can be deduced by the AP(i): there always exists an $N \in \mathbf{N}$ such that $N > \frac{1}{\epsilon}$. The actual proof goes as follows.

Let $\epsilon > 0$ be arbitrary. By the AP(i), there exists an $N \in \mathbf{N}$ such that $N > \frac{1}{\epsilon}$. Then whenever $n \in N$ we have $1/n \leq 1/N < \epsilon$ and hence

$$|a_n - a| = \left|\frac{n+1}{n} - 1\right| = \frac{1}{n} < \epsilon.$$

Therefore, by definition, $\lim a_n = 1$. □

Topological definition of limits. Given a real number $a \in \mathbf{R}$ and positive number $\epsilon > 0$ the interval $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$ is the **ϵ-neighborhood** of a.

Definition 2.3. A topological definition of $\lim a_n = a$ is the following:

$$\forall \epsilon > 0\ \exists N \in \mathbf{N}\ \forall n \in \mathbf{N}\ (n \geq N \Longrightarrow a_n \in V_\epsilon(a)).$$

2.2. The Algebraic and Order Limit Theorems

Definition 2.4. A sequence (a_n) is **bounded above** (or **bounded below**) if there exists a number M such that $a_n \leq M$ (or $a_n \geq M$) for $n \in \mathbf{N}$. A sequence (a_n) is **bounded** if it is both bounded above and bounded above; or, *equivalently,* there exists a number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$.

Theorem 2.1. *Every convergent sequence is bounded.*

Proof. Let $(a_n) \to a$. Then there exists an $N \in \mathbf{N}$ such that

$$|a_n - a| < 1 \quad \forall\ n \geq N.$$

Hence, by the **Triangle Inequality**, $|a_n| = |(a_n - a) + a| \leq |a_n - a| + |a| \leq 1 + |a|$ for all $n \geq N$. Now let

$$M = \max\{|a_1|, |a_2|, \cdots, |a_{N-1}|, |a| + 1\}.$$

Then $|a_n| \leq M$ for all $n \in N$. This proves (a_n) is bounded. □

Theorem 2.2 (**Algebraic Limit Theorem**). *Let* $\lim a_n = a$ *and* $\lim b_n = b$. *Then*

(i) $\lim(ca_n) = ca$ *for all* $c \in \mathbf{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$, *provided* $b_n \neq 0$ *and* $b \neq 0$.

Warning: *We can use these formulas only when both the limits* $\lim a_n$ *and* $\lim b_n$ *exist.*

Proof. We only include the proof for the product and quotient theorem.
Proof of (iii): Note that

$$a_n b_n - ab = a_n b_n - a_n b + a_n b - ab = a_n(b_n - b) + (a_n - a)b.$$

Therefore, by the Triangle Inequality,

$$|a_n b_n - ab| \leq |a_n(b_n - b)| + |(a_n - a)b| = |a_n||b_n - b| + |a_n - a||b|.$$

Given $\epsilon > 0$, in order to make $|a_n b_n - ab| < \epsilon$, it suffices to make each of the two terms on the righthand side $< \epsilon/2$. Since (a_n) converges, it is bounded and so $|a_n| \leq M$ ($\forall\ n \in \mathbf{N}$) for some number $M > 0$. Hence the two terms are bounded as follows:

$$|a_n||b_n - b| \leq M|b_n - b|, \quad |a_n - a||b| \leq |a_n - a|(|b| + 1)$$

(here we change $|b| \geq 0$ to $|b| + 1 > 0$ to divide later). Now, given arbitrary $\epsilon > 0$, since $(a_n) \to a$, we have $N_1 \in \mathbf{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2(|b| + 1)} \quad \forall\, n \geq N_1.$$

Since $(b_n) \to b$, we have $N_2 \in \mathbf{N}$ such that

$$|b_n - b| < \frac{\epsilon}{2M} \quad \forall\, n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$ (or $N = N_1 + N_2$). Then, for this N, whenever $n \geq N$, it follows that

$$|a_n - a| < \frac{\epsilon}{2(|b| + 1)}, \quad |b_n - b| < \frac{\epsilon}{2M};$$

hence

$$|a_n - a||b| \leq \frac{\epsilon|b|}{2(|b| + 1)} < \frac{\epsilon}{2}, \quad |a_n||b_n - b| \leq M|b_n - b| < \frac{\epsilon}{2},$$

and finally, it follows that, whenever $n \geq N$,

$$|a_n b_n - ab| \leq |a_n||b_n - b| + |a_n - a||b| < \epsilon/2 + \epsilon/2 = \epsilon.$$

By definition, $(a_n b_n) \to ab$.

Proof of (iv): Note that

$$\frac{a_n}{b_n} - \frac{a}{b} = \frac{ba_n - ab_n}{b_n b} = \frac{b(a_n - a) + a(b - b_n)}{b_n b}.$$

Hence

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \le \frac{|b||a_n - a|}{|b_n b|} + \frac{|a||b - b_n|}{|b_n b|}. \tag{2.1}$$

Since $(b_n) \to b \neq 0$, with $\epsilon = |b|/2 > 0$, there exists an $N_1 \in N$ such that $|b_n - b| < |b|/2$ for all $n \ge N_1$. Hence, by the triangle inequality, $|b_n| \ge |b| - |b_n - b| \ge |b|/2$ for all $n \ge N_1$. So, for all $n \ge N_1$, we have $|b_n b| \ge |b|^2/2$ and hence

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \le \frac{|b||a_n - a|}{|b_n b|} + \frac{|a||b - b_n|}{|b_n b|}$$

$$\le \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b| \le \frac{2}{|b|}|a_n - a| + \frac{2|a|+1}{|b|^2}|b_n - b|.$$

We then use the convergences as before to select N_2 and N_3 in $\mathbf{N}$ such that

$$|a_n - a| < \frac{\epsilon|b|}{4} \quad \text{whenever } n \ge N_2$$

and

$$|b_n - b| < \frac{\epsilon|b|^2}{2(2|a|+1)} \quad \text{whenever } n \ge N_3.$$

Finally, let $N = \max\{N_1, N_2, N_3\}$. Then, whenever $n \ge N$, it follows that

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \frac{2}{|b|}|a_n - a| + \frac{2|a|+1}{|b|^2}|b_n - b| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Theorem 2.3 (Order Limit Theorem). *Assume* $\lim a_n = a$ *and* $\lim b_n = b$ *both exist. If* $a_n \le b_n$ *for all* $n \ge N_0$*, where* $N_0 \in \mathbf{N}$ *is some number, then* $a \le b$.

Proof. We use the proof by contradiction. Suppose $a > b$. Then $\lim(a_n - b_n) = a - b > 0$. Using $\epsilon = \frac{a-b}{2} > 0$, we have an $N \in \mathbf{N}$ such that

$$|(a_n - b_n) - (a - b)| < \epsilon = \frac{a-b}{2} \quad \forall\, n \ge N.$$

Hence $a - b - \epsilon < a_n - b_n < a - b + \epsilon$ for all $n \ge N$. But $a - b - \epsilon = \frac{a-b}{2} > 0$; this implies that $a_n - b_n > a - b - \epsilon > 0$ for all $n \ge N$. So $a_n > b_n$ for all $n \ge N$; in particular, $a_n > b_n$ for $n = N_0 + N > N_0$, which contradicts the assumption $a_n \le b_n$ for all $n \ge N_0$. □

Example 2.4. Let $x_n \ge 0$ for all $n \in \mathbf{N}$ and $\lim(x_n) = x$. Show $\lim(\sqrt{x_n}) = \sqrt{x}$.

Proof. We must have $x \ge 0$ by the order limit theorem. Prove the statement in the following two cases:

Case 1: $x = 0$. Note that $\sqrt{x_n} < \epsilon$ if and only if $x_n < \epsilon^2$.

Case 2: $x > 0$. In this case note that

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \le \frac{|x_n - x|}{\sqrt{x}}.$$

Details are left as a homework! □

2.3. The Monotone Convergence Theorem and a First Look at Infinite Series

Definition 2.5. A sequence (a_n) is called **increasing** if $a_n \le a_{n+1}$ for all $n \in \mathbf{N}$, and is called **decreasing** if $a_n \ge a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is said to be **monotone** if it is either increasing or decreasing.

Theorem 2.4 (**Monotone Convergence Theorem** (**MCT**))**.** *If an increasing (or decreasing) sequence is bounded above (or below) then it converges. In fact, the limit equals the supremum (or the infimum) of the set consisting of the values of the sequence.*

Proof. We only prove the theorem for increasing sequences. Let (a_n) be an increasing sequence and, for some number M, $a_n \le M$ for all $n \in \mathbf{N}$. Consider the set $S = \{a_n : n \in \mathbf{N}\}$. Then S is nonempty and bounded above (with M being an upper-bound). So by the (AoC), $a = \sup S$ exists. We prove $\lim a_n = a$. Since a is an upper-bound for S, we have

$$a_n \le a \quad \forall\, n \in \mathbf{N}.$$

On the other hand, given arbitrary $\epsilon > 0$, since $a = \sup S$, by the Lemma before, there exists an element $a_N \in S$ such that $a - \epsilon < a_N$. Then, by the monotonicity of a_n,

$$a_n \ge a_N > a - \epsilon \quad \forall\, n \ge N.$$

Combining above inequalities, we have $a - \epsilon < a_n \le a < a_n + \epsilon$; that is, $|a_n - a| < \epsilon$ for all $n \ge N$. Hence, by definition, $\lim a_n = a$. □

The MCT is useful for determining the convergence of a sequence without explicitly knowing the actual limit and checking the ϵ-N definition.

Example 2.5. Show that the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

converges and find the limit.

Solution. Let a_n be the n-th term of this sequence; that is, $a_1 = \sqrt{2}, a_2 = \sqrt{2\sqrt{2}}, \cdots$. We have

$$a_{n+1} = \sqrt{2a_n}; \quad \text{hence} \quad a_{n+1}^2 = 2a_n, \quad \forall\, n = 1, 2, 3, \cdots .$$

Use induction and we can show that

$$\sqrt{2} \le a_n \le 2 \text{ and } a_n \le a_{n+1} \text{ for all } n \in \mathbf{N}.$$

Hence (a_n) is bounded and increasing. Therefore, by the MCT, $\lim a_n = a$ exists. Moreover. the order limit theorem says $\sqrt{2} \le a \le 2$. By an easy check using the ϵ-N definition, we have that $\lim(a_{n+1}) = a$. So, taking the limit on both sides of $a_{n+1}^2 = 2a_n$, we have $a^2 = 2a$. Since $a \ne 0$, it follows that $a = 2$; that is, $\lim a_n = 2$. □

Limit Superior and Limit Inferior*. (This is covered in an Exercise.) Let (a_n) be a sequence. Let

$$x_n = \inf\{a_k \,|\, k \ge n\} = \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\},$$
$$y_n = \sup\{a_k \,|\, k \ge n\} = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\}.$$

Then, if (a_n) is bounded above then (x_n) is increasing and bounded, and if (a_n) is bounded below then (y_n) is decreasing and bounded. Hence, by the **MCT,** the limits $\lim x_n = x$ and $\lim y_n = y$ exist, respectively when (a_n) is bounded above or bounded below.

Definition 2.6 (Limit superior and limit inferior)**.** Let (a_n) be bounded. Define

$$\liminf a_n = \lim x_n = x, \quad \limsup a_n = \lim y_n = y$$

to be the **limit inferior** and **limit superior** of the sequence (a_n), respectively.

Theorem 2.5. *One has that* $\liminf a_n \le \limsup a_n$. *Furthermore,* $\lim a_n$ *exists if and only if* $\liminf a_n = \limsup a_n$. *In this case* $\lim a_n = \liminf a_n = \limsup a_n$.

Proof. 1. The first statement is easy from the **order limit theorem** and the **inequality**

$$x_n \le a_n \le y_n \quad \forall\, n \in \mathbf{N}, \tag{2.2}$$

where x_n, y_n are defined as above.

2. Assume $\liminf a_n = \limsup a_n = l$; hence $\lim x_n = \lim y_n = l$. Then, from (2.2) and the **Squeeze Theorem,** $\lim a_n = l$.

3. Assume $\liminf a_n \ne \limsup a_n$. Take two numbers a, b such that $\liminf a_n < a < b < \limsup a_n$. Since $\lim x_n < a$ and x_n is increasing, it follows that $x_k < a\ \forall\, k \in \mathbf{N}$. So, as $x_k = \inf\{a_n \mid n \ge k\}$, $\forall\, k \in \mathbf{N}$, $\exists\, n_k \ge k$ such that $a_{n_k} < a$. Similarly, $\forall\, k \in \mathbf{N}$, $\exists\, m_k \ge k$ such that $a_{m_k} > b$. Hence

$$a_{m_k} - a_{n_k} > b - a \quad \forall\, k \in \mathbf{N}. \tag{2.3}$$

We show that (a_n) does not converge. For a contradiction, suppose that $(a_n) \to l$. Then, for $\epsilon = \frac{b-a}{4} > 0$, $\exists\, N \in \mathbf{N}$ such that

$$|a_n - l| < (b-a)/4 \quad \forall\, n \ge N.$$

Let $k \ge N$; then m_k, $n_k \ge k \ge N$. Hence $|a_{m_k} - l| < \frac{b-a}{4}$ and $|a_{n_k} - l| < \frac{b-a}{4}$. This implies

$$|a_{m_k} - a_{n_k}| \le |a_{m_k} - l| + |a_{n_k} - l| < (b-a)/2,$$

contradicting with (2.3) above. This completes the proof. □

A First Look at Infinite Series. Let (b_n) be a sequence. An **infinite series** of (b_n) is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots .$$

The corresponding **sequence of partial sums** (s_n) is defined by

$$s_n = b_1 + b_2 + \cdots + b_n \quad \forall\, n \in \mathbf{N}.$$

Definition 2.7. We say that the infinite series $\sum_{n=1}^{\infty} b_n$ **converges** (to $B \in \mathbf{R}$) if $\lim s_n = B$. In this case, we write $\sum_{n=1}^{\infty} b_n = B$ and call the limit B the **value** or the **sum of the infinite series**.

If the partial sum sequence (b_n) diverges, then we say that the infinite series $\sum_{n=1}^{\infty} b_n$ **diverges**.

Infinite Series with Nonnegative Terms. Note that if $b_n \geq 0$ then its partial sum sequence (s_n) is increasing. Therefore, in this case, to show the series to converge, by the MCT, it suffices to show that its partial sum sequence (s_n) is *bounded above.*

EXAMPLE 2.6. (i) Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Note that

$$b_n = \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} \quad \forall\, n \geq 2.$$

Using the **telescoping** technique, we have

$$s_n = b_1 + b_2 + \cdots + b_n < 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n} < 2.$$

So the partial sum sequence (s_n) is bounded above; hence (s_n) converges. Therefore, by definition, the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(ii) Consider the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$. We show that this series diverges by showing that its partial sum sequence (s_n) is not bounded, where

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Note that $s_2 = 1 + \frac{1}{2} = \frac{3}{2}$ and

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = \frac{4}{2}.$$

We can examine s_8 to see $s_8 > \frac{5}{2}$. In general, examine s_{2^k} and we find that

$$\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + \cdots + 2^{k-1} \times \frac{1}{2^k} = 1 + k \times \frac{1}{2} = \frac{k+2}{2}
\end{aligned}$$

for all $k \in \mathbf{N}$. Consequently, (s_n) can not be bounded above by any number. (**Why?**) This proves that the harmonic series diverges.

The results in the previous example are a special case of a general Cauchy's theorem.

Theorem 2.6 (**Cauchy Condensation Test**). *Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots$$

converges.

Proof. It suffices to prove that the partial sum sequence of $\sum a_n$ is bounded above if and only if the partial sum sequence of $\sum 2^k a_{2^k}$ is bounded above.

Let $s_n = a_1 + a_2 + \cdots + a_n$ and

$$t_k = \sum_{j=0}^{k} 2^j a_{2^j} = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}.$$

For any $n \in \mathbf{N}$, if $2^k > n$, then

$$s_n \le a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \le a_1 + 2a_2 + \cdots + 2^k a_{2^k} = t_k.$$

Hence, if $\{t_k\}$ is bounded above, then so is $\{s_n\}$.

For any $k \in \mathbf{N}$, if $n > 2^k$, then

$$\begin{aligned} s_n &\ge a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\ge \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1}a_{2^k} = \frac{1}{2}t_k, \end{aligned}$$

so that $t_k \le 2s_n$, and hence if $\{s_n\}$ is bounded above, then so is $\{t_k\}$. □

Theorem 2.7. *The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.*

Proof. Clearly, if $p \le 0$, then $\frac{1}{n^p} \ge 1$ and hence $\sum \frac{1}{n^p}$ diverges. If $p > 0$ then Theorem 2.6 is applicable, and we are led to the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k},$$

which is a **geometric series** with ratio $x = 2^{1-p}$; hence the result follows from the convergence of geometric series discussed later. □

2.4. Subsequences and the Bolzano-Weierstrass Theorem

We know that every convergent sequence is bounded; however, a bounded sequence may not be convergent, e.g., $((-1)^n)$ is a bounded sequence, but diverges. But, part of the sequence consists of only number 1 and, as a sequence itself, does converge. This is in fact valid for all bounded sequences. We need to introduce the concept of *subsequences.*

Definition 2.8. Let (a_n) be a sequence, and let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_k})_{k=1}^{\infty} = (a_{n_1}, a_{n_2}, a_{n_3}, \cdots)$$

is called a **subsequence** of (a_n). Note that the order of the terms in a subsequence is kept unchanged as in the original sequence.

Theorem 2.8. *Any subsequence of a convergent sequence converges to the same limit as the original convergent sequence.*

Proof. Let $(a_n) \to a$ and $(a_{n_k})_{k=1}^{\infty}$ be a subsequence of (a_n). Given each $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that $|a_n - a| < \epsilon$ for all $n \ge N$. Now we look at all terms a_{n_k} with $k \ge N$. Since (n_k) is a strictly increasing sequence of natural numbers, we must have $n_k \ge k$ for all $k \in \mathbf{N}$. Hence, $\forall k \ge N$, we have $n_k \ge k \ge N$ and thus $|a_{n_k} - a| < \epsilon$; this implies, as a sequence itself, $(a_{n_k}) \to a$ (as $k \to \infty$). □

Remark 2.9. (i) If a sequence has two subsequences converging to two distinct limits, then the original sequence must diverge.

(ii) If a sequence converges, then we may use some of its subsequence to find the limit.

Example 2.7. Let $0 < b < 1$. Show that $(b^n) \to 0$.

Proof. Note that

$$b > b^2 > b^3 > b^4 > \cdots > 0.$$

Hence the sequence $(a_n) = (b^n)$ is a decreasing sequence and bounded below by 0. By the MCT, $s = \lim(b^n)$ exists. By the order theorem $0 \le s \le b < 1$. We consider its subsequence $(a_{2n}) = (b^{2n})$, by the theorem above, $\lim(a_{2n}) = s$. However, $a_{2n} = b^{2n} = a_n \cdot a_n$ and hence, by the product rule,

$$s = \lim(a_{2n}) = \lim(a_n \cdot a_n) = \lim(a_n) \cdot \lim(a_n) = s \cdot s = s^2.$$

Hence $s(1-s) = 0$; but $s < 1$, so we must have $s = 0$ and thus $(b^n) \to 0$. □

Theorem 2.9 (**Bolzano-Weierstrass Theorem** (**BW**))**.** *Every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence; namely there exists a number $M > 0$ such that $|a_n| \le M$ for all $n \in \mathbf{N}$. Bisect the closed interval $[-M, M]$ into two closed intervals $[-M, 0]$ and $[0, M]$. Then *at least one* of the two intervals contains an infinite number of the terms of (a_n). Select one such interval and call it I_1. Let a_{n_1} be a term such that $a_{n_1} \in I_1$. Next, bisect I_1 into two closed intervals meeting at the mid-point of I_1. Let I_2 be one such half interval that again contains an infinite number of the terms of (a_n); such an I_2 must exist since I_1 contains an infinite number of the terms of (a_n). There must be a term $a_{n_2} \in I_2$ with $n_2 > n_1$ for, if not, I_2 would contain at most n_1 terms of (a_n). Proceed in this way, and we construct the closed interval I_k to be one such half of I_{k-1} that contains an infinite number of terms of (a_n) and then select a term $a_{n_k} \in I_k$ with $n_k > n_{k-1}$. Then we obtain a subsequence (a_{n_k}) of (a_n). We want to show (a_{n_k}) converges. First, using the NIP, we have a point $a \in I_k$ for all $k \in \mathbf{N}$. Since both a and a_{n_k} are in I_k, it follows that $|a_{n_k} - a| \le$ the length of the interval I_k. But the length of I_k is $M(1/2)^{k-1}$, which converges to 0 by the previous example. Hence, given any $\epsilon > 0$, there exists a $N \in \mathbf{N}$ such that for all $k \ge N$ we have $|a_{n_k} - a| < \epsilon$. This proves $\lim a_{n_k} = a$. The proof is completed. □

2.5. The Cauchy Criterion

Definition 2.10. A sequence (a_n) of real numbers is called a **Cauchy sequence** if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $m, n \ge N$ in $\mathbf{N}$ it follows that $|a_n - a_m| < \epsilon$; that is,

$$\forall\, \epsilon > 0\ \exists\, N \in \mathbf{N}\ \forall\, m, n \in \mathbf{N}\ (m, n \ge N \Longrightarrow |a_m - a_n| < \epsilon).$$

Theorem 2.10. *Every convergent sequence is a Cauchy sequence.*

Proof. Assume $(a_n) \to a$. Then, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n \ge N$ in $\mathbf{N}$ it follows that $|a_n - a| < \epsilon/2$. (Notice here we use $\epsilon/2$ in place of ϵ in the definition.) Hence, whenever $n, m \ge N$ in $\mathbf{N}$, it follows by the triangle inequality that

$$|a_n - a_m| = |(a_n - a) + (a - a_m)| \le |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, by definition, (a_n) is a Cauchy sequence. □

Theorem 2.11. *Every Cauchy sequence is bounded.*

Proof. Let (a_n) be Cauchy. Then there exists an $N \in \mathbf{N}$ such that $|a_n - a_m| < 1$ for all $n, m \in \mathbf{N}$ and $n, m \geq N$. With $m = N$ we have $|a_n - a_N| < 1$ for all $n \geq N$. Hence, by the triangle inequality, $|a_n| \leq |a_n - a_N| + |a_N| \leq |a_N| + 1$ for all $n \geq N$. Let

$$M = \max\{|a_1|, |a_2|, \cdots, |a_{N-1}|, |a_N| + 1\}.$$

Then it follows that $|a_n| \leq M$ for all $n \in \mathbf{N}$. Hence (a_n) is bounded. $\square$

Theorem 2.12 (Cauchy's Criterion (CC)). *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Proof. We have already shown that a convergent sequence is Cauchy. We need to prove that every Cauchy sequence converges. So let (a_n) be a Cauchy sequence. The previous theorem asserts that (a_n) is bounded. Hence, by the BW, (a_n) has a convergent subsequence (a_{n_k}); let $\lim a_{n_k} = a$. We show that the whole sequence (a_n) converges to a. So let $\epsilon > 0$. Since (a_n) is Cauchy, there exists an $N_1 \in \mathbf{N}$ such that

$$|a_m - a_n| < \epsilon/2 \quad \forall\, n, m \geq N_1.$$

Now, as $\lim a_{n_k} = a$, there exists an $N_2 \in \mathbf{N}$ such that

$$|a_{n_k} - a| < \epsilon/2 \quad \forall\, k \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Assume $m \in \mathbf{N}$ and $m \geq N$. Then $m \geq N_2$ and hence

$$|a_{n_m} - a| < \epsilon/2.$$

Also note that since $1 \leq n_1 < n_2 < n_3 < \cdots$ are natural numbers, it follows that $n_k \geq k$ for all $k \in \mathbf{N}$. Hence $n_m \geq m \geq N_1$. So

$$|a_m - a_{n_m}| < \epsilon/2.$$

Hence, by the triangle inequality, we have

$$|a_m - a| \leq |a_m - a_{n_m}| + |a_{n_m} - a| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Consequently, by definition, $(a_m) \to a$. This completes the proof. $\square$

Completeness Revisited*. We have learned **five properties** about the set $\mathbf{R}$ real numbers, namely, AoC, NIP, MCT, BW and CC. We have also proved that $AoC \Longrightarrow MCT$ and that

$$AoC \Longrightarrow NIP \Longrightarrow BW \Longrightarrow CC.$$

Each of these statements has its own way to characterize the fundamental property of $\mathbf{R}$ that we called the **completeness of R**.

The AoC is the start to derive other statements; however, in fact, each of these statements (AoC, NIP, MCT, BW, CC) implies all the other statements. You may try to prove these implications according to an exercise in the book.

2.6. Properties of Infinite Series

Theorem 2.13 (Algebraic Limit Theorems for Series). *If* $\sum_{k=1}^\infty a_k = A$ *and* $\sum_{k=1}^\infty b_k = B$, *then*

$$\sum_{k=1}^\infty (ca_k + db_k) = cA + dB \quad \textit{for all } c, d \in \mathbf{R}.$$

Note that there is no similar rule for the product series $\sum_{k=1}^\infty (a_k b_k)$ or the quotient series $\sum_{k=1}^\infty (a_k/b_k)$.

Theorem 2.14 (**Cauchy Criterion for Series**). *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given any $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Proof. Note that $s_n - s_m = a_{m+1} + a_{m+2} + \cdots + a_n$. □

Theorem 2.15. *If $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.*

Proof. $a_k = s_k - s_{k-1}$. □

This easy result is often used to show a series *diverges* by showing the sequence of terms does not converge to 0. However, it can not be used to show the convergence simply from the limit $(a_k) \to 0$, as seen from the divergent harmonic series.

Theorem 2.16 (**Comparison Test**). *Assume (a_k) and (b_k) are sequences satisfying*

$$b_k \geq a_k \geq 0 \quad \forall\, k \geq N,$$

where $N \in \mathbf{N}$ is some integer.

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. Note that (i) and (ii) are equivalent (each is a contrapositive statement of the other). Statement (i) can be proved easily by the Cauchy Criterion for Series (**write a proof yourself**). □

Remark 2.11. The comparison test gives **no information** on the convergence of a larger series $\sum_{k=1}^{\infty} b_k$ from the convergence of a smaller series $\sum_{k=1}^{\infty} a_k$. In fact, when the smaller series $\sum_{k=1}^{\infty} a_k$ converges, the larger series $\sum_{k=1}^{\infty} b_k$ could either converge or diverge. For example, let $(a_n) = (\frac{1}{n^4})$; then consider $(b_n) = (\frac{1}{n})$ or $(b_n) = (\frac{1}{n^2})$.

Geometric Series. A series of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots \tag{2.4}$$

is called a **geometric series** of ratio r. Note that the series starts with $k = 0$ instead of $k = 1$. If $r \neq 1$ then the partial sum

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r} \quad (r \neq 1), \quad n \in \mathbf{N}.$$

The sequence (s_n) converges if and only if $|r| < 1$. Therefore the geometric series (2.4) converges if and only if its ratio r satisfies $|r| < 1$. In this case, the sum of the geometric series is given by

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1 - r} \quad \text{when } |r| < 1. \tag{2.5}$$

The well-known **ratio test** and **root test**, given in the Exercises, are both based on the comparison with geometric series; we do not state them here.

Theorem 2.17 (**Absolute Convergence Test**). *If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges as well.*

Proof. Use the CC for series with the triangle inequality

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \le |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| \quad \forall\, n > m \ge 1.$$

(**Fill in the details!**) □

Definition 2.12. We say that the series $\sum_{k=1}^{\infty} a_k$ **converges absolutely** if $\sum_{k=1}^{\infty} |a_k|$ converges. We say that the series $\sum_{k=1}^{\infty} a_k$ **converges conditionally** if $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges.

Theorem 2.18 (**Alternating Series Test**). *Let* (a_n) *satisfy*

$$a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0, \quad (a_n) \to 0.$$

Then the **alternating series** $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ *converges.*

Proof. Let (s_n) be the partial sum sequence, where

$$s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots + (-1)^{n+1} a_n.$$

If $n > m \ge 1$, then

$$s_n - s_m = \sum_{k=1}^{n} (-1)^{k+1} a_k - \sum_{k=1}^{m} (-1)^{k+1} a_k$$

$$= \sum_{k=m+1}^{n} (-1)^{k+1} a_k = (-1)^{m+2} (a_{m+1} - a_{m+2} + \cdots + (-1)^{n-m-1} a_n).$$

We claim

$$|s_n - s_m| = |a_{m+1} - a_{m+2} + \cdots + (-1)^{n-m-1} a_n| \le a_{m+1}. \tag{2.6}$$

To prove this claim, let $S_{n,m} = a_{m+1} - a_{m+2} + \cdots + (-1)^{n-m-1} a_n$. Clearly the sum $S_{n,m}$ has $n-m$ terms whose last term $(-1)^{n-m-1} a_n$ equals $-a_n$ if $n-m$ is even and equals $+a_n$ if $n-m$ is odd. By a simple grouping, we have

$$S_{n,m} = \begin{cases} (a_{m+1} - a_{m+2}) + (a_{m+3} - a_{m+4}) + \cdots + (a_{n-1} - a_n) & \text{if } n-m \text{ is even,} \\ (a_{m+1} - a_{m+2}) + (a_{m+3} - a_{m+4}) + \cdots + (a_{n-2} - a_{n-1}) + a_n & \text{if } n-m \text{ is odd;} \end{cases}$$

hence $S_{n,m} \ge 0$. So $|s_n - s_m| = S_{n,m}$. Next, by a different grouping, we have

$$S_{n,m} = \begin{cases} a_{m+1} - (a_{m+2} - a_{m+3}) - \cdots - (a_{n-2} - a_{n-1}) - a_n & \text{if } n-m \text{ is even,} \\ a_{m+1} - (a_{m+2} - a_{m+3}) - \cdots - (a_{n-1} - a_n) & \text{if } n-m \text{ is odd;} \end{cases}$$

therefore, $S_{n,m} \le a_{m+1}$, and thus (2.6) is proved. Finally, since $(a_{m+1}) \to 0$, using (2.6), we deduce that (s_k) is Cauchy and hence the proof is done. □

Other Tests. There are other useful tests (e.g., **Dirichlet's test** and **Abel's test**) that can be proved using the **summation by parts**; see Exercises 2.7.12-14. Later we will prove **Abel's test** in connection with the power series.

EXAMPLE 2.8. (i) The **alternating harmonic series:**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the **alternating series test.** But since the series of absolute values $\sum \frac{1}{n}$ diverges, this alternating series converges **conditionally**.

(ii) The series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$
is also a convergent alternating series, but converges **absolutely** because the absolute series $\sum \frac{1}{n^2}$ converges. So there are two tests we can use to see the convergence of this series; however, the **alternating series test** only asserts the convergence and does not tell whether the convergence is conditional or absolute.

(iii) Often, to determine whether a series converges or not, you should first try to use the **absolute convergence test**; if it does not work, then try to use other tests.

Rearrangements*. Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, we say that $\sum_{n=1}^{\infty} b_n$ is a **rearrangement** of $\sum_{n=1}^{\infty} a_n$ if there exists a *1-1 and onto* function $f\colon \mathbf{N} \to \mathbf{N}$ such that
$$b_n = a_{f(n)} \quad \forall\, n \in \mathbf{N}.$$
Since, using the inverse function f^{-1} of f, we also have $a_n = b_{f^{-1}(n)}$ for all $n \in \mathbf{N}$, we see every term of $\sum_{n=1}^{\infty} b_n$ appears *exactly once* in $\sum_{n=1}^{\infty} a_n$ and vice-versa, every term of $\sum_{n=1}^{\infty} a_n$ appears *exactly once* in $\sum_{n=1}^{\infty} b_n$.

Now, if a series $\sum_{n=1}^{\infty} a_n$ and one of its rearrangements $\sum_{n=1}^{\infty} b_n$ both converge, do we have that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$?

EXAMPLE 2.9. Consider the alternating harmonic series
$$S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2 \quad \text{(see Chapter 6)}.$$
Then
$$\frac{1}{2} S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots + (-1)^{n+1} \frac{1}{2n} + \cdots .$$
So
$$S + \frac{1}{2} S = \frac{3}{2} S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$
becomes a rearrangement of $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$; but certainly they are not equal.

Theorem 2.19. *If $\sum_{n=1}^{\infty} |a_n|$ converges, then, for any rearrangement function $f\colon \mathbf{N} \to \mathbf{N}$, it follows that*
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{f(n)}.$$

Proof. Let $b_k = a_{f(k)}$ for $k \in \mathbf{N}$. For $n, m \in \mathbf{N}$, define
$$s_n = a_1 + a_2 + \cdots + a_n; \quad t_m = b_1 + b_2 + \cdots + b_m.$$
Let $(s_n) \to A$. We show that $(t_m) \to A$. Given any $\epsilon > 0$, we find an $N \in \mathbf{N}$ such that
$$|s_N - A| < \epsilon/2, \qquad \sum_{k=m+1}^{n} |a_k| < \epsilon/2 \quad \forall\, n > m \geq N.$$
Since $f\colon \mathbf{N} \to \mathbf{N}$ is 1-1 and onto, let $\{i_1, i_2, \cdots, i_N\} \subseteq \mathbf{N}$ be such that $f(i_k) = k$ for each $k = 1, 2, \cdots, N$. Let
$$M = \max\{i_1, i_2, \cdots, i_N\}.$$

Then $M \geq N$. Let $m \in \mathbf{N}$ be such that $m \geq M$. Then, since $\{i_1, i_2, \cdots, i_N\} \subseteq \{1, 2, 3, \cdots, m\}$, it follows that

$$\begin{aligned} t_m = b_1 + b_2 + \cdots + b_m &= a_{f(1)} + a_{f(2)} + \cdots + a_{f(m)} \\ &= a_{f(i_1)} + a_{f(i_2)} + \cdots + a_{f(i_N)} + \sum_{j \in J} a_{f(j)} \\ &= a_1 + a_2 + \cdots + a_N + \sum_{j \in J} a_{f(j)} \\ &= S_N + \sum_{j \in J} a_{f(j)}, \end{aligned}$$

where $J = \{1, 2, 3, \cdots, m\} \setminus \{i_1, i_2, \cdots, i_N\}$. Since $J \cap \{i_1, i_2, \cdots, i_N\} = \emptyset$, we have $f(j) \geq N + 1$ for all $j \in J$. Let $K = \max\{f(j) \,:\, j \in J\} \geq N + 1$. Then $N + 1 \leq f(j) \leq K$ for all $j \in J$ and hence

$$\left| \sum_{j \in J} a_{f(j)} \right| \leq \sum_{j \in J} |a_{f(j)}| \leq \sum_{k=N+1}^{K} |a_k| < \epsilon/2.$$

Finally, it follows that, for all $m \geq M$,

$$\begin{aligned} |t_m - A| &\leq |S_N - A| + \left| \sum_{j \in J} a_{f(j)} \right| \\ &< \epsilon/2 + \sum_{j \in J} |a_{f(j)}| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This proves $(t_m) \to A$. $\square$