Chapter 5

The Derivative

5.1. Derivatives and the Intermediate Value Property

Definition 5.1. Let $f\colon (a,b) \to \mathbf{R}$ be a function and $c \in (a,b)$. We say f is **differentiable at** c if the limit

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

exists. In this case, the limit is called the **derivative** of f at c and is denoted by $f'(c)$.

We say f is differentiable in (a,b) if it is differentiable at every point of (a,b).

Example 5.1. (a) $f(x) = x^n$ ($n \in \mathbf{N}$) is differentiable at every $c \in \mathbf{R}$.

(b) $g(x) = |x|$ is not differentiable at $x = 0$.

Theorem 5.1. *If $f\colon (a,b) \to \mathbf{R}$ is differentiable at a point $c \in (a,b)$, then f is continuous at c.*

Proof. Use

$$f(x) - f(c) = \frac{f(x)-f(c)}{x-c} \cdot (x-c).$$

$\square$

Combinations of Differentiable Functions.

Theorem 5.2 (**Algebraic Differentiation Rules**). *Let $f, g\colon (a,b) \to \mathbf{R}$ be both differentiable at some point $c \in (a,b)$. Then so are functions $kf + lg$ ($k, l \in \mathbf{R}$), fg and f/g (if $g(c) \neq 0$) differentiable at c. Moreover, it follows that*

(i) $(kf + lg)'(c) = kf'(c) + lg'(c)$,

(ii) **Product Rule**: $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, *and*

(iii) **Quotient Rule**: $(f/g)'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$, *provided* $g(c) \neq 0$.

Proof. We prove (ii) only. Note that

$$\begin{aligned} f(x)g(x) - f(c)g(c) &= f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c) \\ &= f(x)[g(x) - g(c)] + g(c)[f(x) - f(c)]. \end{aligned}$$

Hence
$$\frac{f(x)g(x)-f(c)g(c)}{x-c} = f(x)\cdot\frac{g(x)-g(c)}{x-c} + g(c)\cdot\frac{f(x)-f(c)}{x-c}.$$
Since the two terms on the right-hand side have limits as $x \to c$ and the limits can be found by the algebraic limit theorem, it follows that the limit
$$\lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c} = (fg)'(c)$$
also exists and equals
$$\begin{aligned}\lim_{x\to c} f(x)\cdot\frac{g(x)-g(c)}{x-c} &+ \lim_{x\to c} g(c)\cdot\frac{f(x)-f(c)}{x-c}\\ &= f(c)g'(c)+g(c)f'(c).\end{aligned}$$
This proves (ii). □

Theorem 5.3 (**Chain Rule**). *Let $f\colon (a,b)\to \mathbf{R}$ and $g\colon (\alpha,\beta)\to\mathbf{R}$ satisfy $f(a,b)\subseteq(\alpha,\beta)$ so that the composition $g\circ f\colon (a,b)\to\mathbf{R}$ is well-defined. If f is differentiable at $c\in(a,b)$ and g is differentiable at $f(c)\in(\alpha,\beta)$, then $g\circ f$ is differentiable at c with derivative given by $(g\circ f)'(c) = g'(f(c))f'(c)$.*

Proof. Define a function $d\colon (\alpha,\beta)\to\mathbf{R}$ by
$$d(y) = \begin{cases} \frac{g(y)-g(f(c))}{y-f(c)} - g'(f(c)) & y\neq f(c)\\ 0 & y = f(c).\end{cases}$$
Note that $\lim_{y\to f(c)} d(y) = 0 = d(f(c))$; hence $d\colon(\alpha,\beta)\to\mathbf{R}$ is continuous at $f(c)$. Using $d(y)$, we write
$$g(y)-g(f(c)) = [g'(f(c))+d(y)](y-f(c)) \quad \forall\ y\in(\alpha,\beta).$$
Hence we have, for all $t\in(a,b)$ and $t\neq c$, that
$$\frac{g(f(t))-g(f(c))}{t-c} = [g'(f(c))+d(f(t))]\frac{f(t)-f(c)}{t-c}.$$
The two functions of t on the right-hand side both have limits as $t\to c$ and hence the function on the left-hand side has limit, given by
$$\begin{aligned}\lim_{t\to c}\frac{g(f(t))-g(f(c))}{t-c} &= \lim_{t\to c}[g'(f(c))+d(f(t))]\lim_{t\to c}\frac{f(t)-f(c)}{t-c}\\ &= g'(f(c))f'(c).\end{aligned}$$
□

Remark. If we do not introduce the function $d(y)$ as above, we might try to directly write
$$\frac{g(f(t))-g(f(c))}{t-c} = \frac{g(f(t))-g(f(c))}{f(t)-f(c)}\cdot\frac{f(t)-f(c)}{t-c}.$$
But the first quotient on the right-hand side may very well be undefined since $f(t)$ may be equal to $f(c)$ even when $t\neq c$.

We have the following familiar first-derivative test in Calculus.

Theorem 5.4 (**Interior Extremum Theorem**). *Let f be differentiable on (a,b). If f attains a relative extremum at a point $c\in(a,b)$, then $f'(c)=0$.*

Proof. We prove the case of a relative maximum. Assume f has a relative maximum at $c \in (a,b)$; namely, there exists a neighborhood $V_\epsilon(c) \subseteq (a,b)$ such that $f(x) \leq f(c)$ for all $x \in V_\epsilon(c)$. We define two sequences (x_n) and (y_n) in $V_\epsilon(c)$ such that $x_n < c < y_n$ for all $n \in \mathbf{N}$ and $(x_n) \to c$ and $(y_n) \to c$. Hence

$$f'(c) = \lim \left(\frac{f(x_n) - f(c)}{x_n - c} \right) = \lim \left(\frac{f(y_n) - f(c)}{y_n - c} \right).$$

But since $x_n < c$ and $f(x_n) \leq f(c)$ we have $\frac{f(x_n)-f(c)}{x_n-c} \geq 0$; similarly, $\frac{f(y_n)-f(c)}{y_n-c} \leq 0$. Hence the order limit theorem for sequences implies $0 \leq f'(c) \leq 0$; so $f'(c) = 0$. $\square$

Intermediate Value Property.

Definition 5.2. A function f has the **intermediate value property** on interval $[a,b]$ if for all $x < y$ in $[a,b]$ with $f(x) \neq f(y)$ and for each number L between $f(x)$ and $f(y)$ there exists a number $c \in (x,y)$ such that $f(c) = L$.

We prove derivative functions have this property.

Theorem 5.5 (**Darboux's Theorem**). *Let f be differentiable on an open interval containing interval $[a,b]$ and $f'(a) \neq f'(b)$. If $\alpha \in \mathbf{R}$ is a number between $f'(a)$ and $f'(b)$, then there exists a point $c \in (a,b)$ such that $f'(c) = \alpha$.*

Proof. We only prove the case of $f'(a) < \alpha < f'(b)$; the case of $f'(b) < \alpha < f'(a)$ is completely similar. Define $g(x) = f(x) - \alpha x$. Then g is differentiable on the interval containing $[a,b]$. Hence $g'(x) = f'(x) - \alpha$ and $g'(a) < 0 < g'(b)$. By **Extreme Value Theorem** for continuous functions, let $c \in [a,b]$ be such that $g(c) = \min_{x\in[a,b]} g(x)$. Show that $c \neq a$ from $g'(a) < 0$ and that $c \neq b$ from $g'(b) > 0$ (**Exercise!**). So $c \in (a,b)$ is an interior extremum point; by the previous theorem, $g'(c) = 0$, which proves $f'(c) = \alpha$. $\square$

5.2. The Mean Value Theorem

Theorem 5.6 (**Rolle's Theorem**). *Let $f\colon [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. Since f is continuous on $[a,b]$, f attains its maximum and minimum on $[a,b]$. If these two extrema are equal, then the function is constant and hence $f'(x) = 0$ for all $x \in (a,b)$. Now assume the two extrema are not equal. Then at least one of them is not attained at the endpoints since two endpoints give the same value. Hence there must be one interior extremum point $c \in (a,b)$, where $f'(c) = 0$. $\square$

Theorem 5.7 (**Mean Value Theorem**). *Let $f\colon [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Apply Rolle's Theorem to the function

$$g(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a) \right].$$

$\square$

Theorem 5.8 (**Generalized Mean Value Theorem**). *Let f, $g\colon [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If $g'(x)$ is never zero on (a,b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Apply the Mean Value Theorem to the function

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Note that the **MVT** is a special case of **Generalized MVT** with $g(x) = x$. □

L'Hospital's Rules.

Theorem 5.9. *Assume f and g are continuous functions defined on an interval containing c and $f(c) = g(c) = 0$, and assume f and g are differentiable on this interval, with the possible exception of the point c. Then*

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L \quad \textit{implies} \quad \lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

Proof. Use the Generalized MVT. □

Infinity Limits.

Definition 5.3. Given $g\colon A \to \mathbf{R}$ and a limit point c of A, we say that $\lim_{x \to c} g(x) = \infty$ if, for every $M > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $g(x) \geq M$. We say that $\lim_{x \to c} g(x) = -\infty$ if $\lim_{x \to c}(-g(x)) = \infty$.

Theorem 5.10. *Assume f and g are differentiable on (a,b), and $\lim_{x \to a} g(x) = \infty$. Then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \quad \textit{implies} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof. Because $\lim_{x \to a} f'(x)/g'(x) = L$, there exists a $\delta_1 \in (0, b-a)$ such that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\epsilon}{2}$$

for all $a < x < a + \delta_1 < b$. Let $t = a + \delta_1$. For any $x \in (a,t)$, we apply the Generalized MVT on $[x,t]$ to get

$$\frac{f(x) - f(t)}{g(x) - g(t)} = \frac{f'(c)}{g'(c)}$$

for some $c \in (x,t)$. Our choice of $\delta_1 > 0$ implies

$$L - \frac{\epsilon}{2} < \frac{f(x) - f(t)}{g(x) - g(t)} < L + \frac{\epsilon}{2} \quad \forall\, x \in (a,t). \tag{5.1}$$

Since $\lim_{x \to a} g(x) = \infty$, there exists a $\delta_2 \in (0, b-a)$ such that $g(x) \geq M = \max\{1, g(t) + 1\}$ for all $a < x < a + \delta_2 < b$. Therefore $1 - \frac{g(t)}{g(x)} \geq \frac{1}{g(x)} > 0$. We multiply (5.1) by the positive term $1 - \frac{g(t)}{g(x)}$ to get

$$\left(L - \frac{\epsilon}{2}\right)\left(1 - \frac{g(t)}{g(x)}\right) < \frac{f(x) - f(t)}{g(x)} < \left(L + \frac{\epsilon}{2}\right)\left(1 - \frac{g(t)}{g(x)}\right),$$

which implies

$$L - \frac{\epsilon}{2} + \frac{-Lg(t) + \frac{\epsilon}{2}g(t) + f(t)}{g(x)} < \frac{f(x)}{g(x)} < L + \frac{\epsilon}{2} + \frac{Lg(t) - \frac{\epsilon}{2}g(t) + f(t)}{g(x)}.$$

This inequality holds for all $x \in (a, a + \delta')$, where $\delta' = \min\{\delta_1, \delta_2\} > 0$. Again, since $\lim_{x \to a} g(x) = \infty$, there exists a $\delta_3 > 0$ such that

$$g(x) \geq \frac{2}{\epsilon}\left(|Lg(t)| + \frac{\epsilon}{2}|g(t)| + |f(t)|\right)$$

for all $x \in (a, a + \delta_3)$. Finally let $\delta = \min\{\delta', \delta_3\} > 0$. Then whenever $x \in (a, a + \delta)$, it follows that

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon.$$

$\square$