Chapter 6

Sequences and Series of Functions

6.1. Convergence of a Sequence of Functions

Pointwise Convergence.

Definition 6.1. Let, for each $n \in \mathbf{N}$, function $f_n \colon A \to \mathbf{R}$ be defined. If, for each $x \in A$, the sequence $(f_n(x))$ converges (to a limit $f(x)$); that is,

$$\lim f_n(x) = f(x) \quad \forall\ x \in A,$$

then we say that (f_n) **converges pointwise** to the limit function f on A. In this case, we write $f_n(x) \to f(x)$ or $f_n \to f$ pointwise on A, or $\lim_{n\to\infty} f_n(x) = f(x)$ to emphasize the limit is for $n \to \infty$.

The pointwise convergence on A means that,

$$\forall\, x \in A\ \forall\, \epsilon > 0\ \exists\, N \in \mathbf{N}\ \forall\, n \in \mathbf{N}\ \ (n \geq N \Longrightarrow |f_n(x) - f(x)| < \epsilon).$$

Note that the number N here depends on both x and ϵ.

Example 6.1. (i) Let $f_n(x) = \frac{x^2+nx}{n}$, $x \in \mathbf{R}$. For any given $x \in \mathbf{R}$, $f_n(x) = x + \frac{x^2}{n} \to x$ as $n \to \infty$. Hence $f_n(x) \to f(x) = x$ pointwise on $\mathbf{R}$.

(ii) Let $g_n(x) = x^n$ on $[0,1]$. Note that $g_n(1) = 1$ for all $n \in \mathbf{N}$; so $(g_n(1)) \to 1$. If $0 \leq x < 1$, then $(g_n(x)) = (x^n) \to 0$. Hence the pointwise limit function of $g_n(x)$ on $[0,1]$ is given by

$$g(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1. \end{cases}$$

Note that although each g_n is continuous on $[0,1]$, the pointwise limit function g is not continuous at $x = 1$.

(iii) Let $h_n(x) = x^{1+\frac{1}{2n-1}}$ on $[-1,1]$. Then $(h_n(x)) \to |x|$ pointwise on $[-1,1]$.

(iv) Let

$$f_n(x) = \begin{cases} \frac{1}{|x|} & \frac{1}{n} \leq |x| \leq 5 \\ n^2|x| & |x| < \frac{1}{n}. \end{cases}$$

Then $f_n(x) \to \frac{1}{|x|}$ $(x \neq 0)$ and $f_n(0) \to 0$. The limit function on $[-5,5]$ is unbounded.

Remark 6.2. From Example (ii) above, we see that

$$\lim_{x\to 1^-}\left(\lim_{n\to\infty} g_n(x)\right) = 0, \quad \lim_{n\to\infty}\left(\lim_{x\to 1^-} g_n(x)\right) = 1.$$

Hence in general

$$\lim_{x\to c}\left(\lim_{n\to\infty} f_n(x)\right) \neq \lim_{n\to\infty}\left(\lim_{x\to c} f_n(x)\right). \tag{6.1}$$

In order to guarantee the equality, we need a convergence stronger than the pointwise convergence.

Definition 6.3. (**Uniform Convergence**) Let $f_n\colon A \to \mathbf{R}$ and $f\colon A \to \mathbf{R}$ be given functions. We say that the sequence (f_n) **converges uniformly on** A to function f if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $x \in A$ and $n \geq N$ it follows that $|f_n(x) - f(x)| < \epsilon$.

Therefore the uniform convergence on A means that

$$\forall\, \epsilon > 0\, \exists\, N \in \mathbf{N}\, \forall\, x \in A\, \forall\, n \in \mathbf{N}\ \ (n \geq N \Longrightarrow |f_n(x) - f(x)| < \epsilon).$$

Note that the number N here depends only on ϵ, not on $x \in A$; the same N works for all $x \in A$.

Remark 6.4. 1. For pointwise convergence, given $\epsilon > 0$, the number N is to be found after $x \in A$ is given (so N depends on x), while for the uniform convergence, the number N is to be found that works for *every* $x \in A$ (so N is independent of x).

2. From the definitions, if (f_n) converges uniformly to f on A then $(f_n(x)) \to f(x)$ pointwise on A. Therefore, the uniform limit function must be the pointwise limit function.

Example 6.2. Consider $f_n(x) = \frac{x^2+nx}{n}$ and $f(x) = x$ on $\mathbf{R}$. We know $(f_n(x)) \to f(x)$ pointwise on $\mathbf{R}$. However, given $\epsilon > 0$, can we find an $N \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| = \frac{x^2}{n} < \epsilon \quad \forall\, n \geq N,\ x \in \mathbf{R}?$$

If such an N existed, we would take $x = \sqrt{N}$ and $n = N$ to obtain $1 < \epsilon$, a contradiction if our ϵ is chosen < 1. Therefore, the sequence (f_n) *does not* converge uniformly to f on $\mathbf{R}$.

In general, by negating the definition, a sequence $(f_n(x))$ does not converge uniformly on A to a function $f(x)$ if and only if there exist number $\epsilon_0 > 0$, sequence (n_k) in $\mathbf{N}$ with $n_{k+1} > n_k$, and sequence (x_k) in A such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0.$$

Example 6.3. We also consider $f_n(x) = \frac{x^2+nx}{n}$ and $f(x) = x$ but on bounded interval $[-b, b]$. Since

$$|f_n(x) - f(x)| = \frac{x^2}{n} \leq \frac{b^2}{n}$$

holds for all $x \in [-b, b]$, in order to make this quantity $< \epsilon$ for all such x', we can choose $N \in \mathbf{N}$ such that $N > \frac{b^2}{\epsilon}$. Then, for all $n \geq N$ and $x \in [-b, b]$, it does follow that $|f_n(x) - f(x)| \leq \frac{b^2}{n} < \epsilon$. Hence (f_n) uniformly converges to f on $[-b, b]$ (but not on whole $\mathbf{R}$, as seen above).

Example 6.4. Show that the convergence $(x^n) \to g(x) = 0$ is not uniform on $[0, 1)$ but is uniform on $[0, b]$ for each $0 < b < 1$.

Proof. Let $g_n(x) = x^n$. We choose $n_k = k$ and $x_k = (1/2)^{1/k} \in (0,1)$. Then $g_k(x_k) = x_k^k = 1/2$. So for $\epsilon_0 = 1/2 > 0$, there cannot be an $N \in \mathbf{N}$ such that $|g_n(x) - 0| < \epsilon_0$ holds for all $n \geq N$ and $x \in [0,1)$ because $g_N(x_N) = 1/2 = \epsilon_0$.

However, if we consider g_n on $[0,b]$ for a given $b \in (0,1)$, then, for each $\epsilon > 0$, since $(b^n) \to 0$, there exists an $N \in N$ such that $b^n < \epsilon$ for all $n \in \mathbf{N}$, $n \geq N$. Then, for all $n \in \mathbf{N}$, $n \geq N$ and $x \in [0,b]$,

$$|g_n(x) - 0| = x^n \leq b^n < \epsilon.$$

This proves that $(g_n(x)) \to 0$ uniformly on $[0,b]$. □

Limit of Uniform Limit Function.

Theorem 6.1 (Limit of uniform limit function). *Let (f_n) and f be functions on A and let (f_n) converge uniformly to f on A. Assume c is a limit point of A and, for each $n \in \mathbf{N}$, $\lim_{x\to c} f_n(x) = a_n$ exists; furthermore, assume $\lim a_n = L$ exists. Then $\lim_{x\to c} f(x) = L$. That is, in this case, we have*

$$\lim_{x\to c}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to c} f_n(x)\right). \tag{6.2}$$

Proof. Given $\epsilon > 0$, by the uniform convergence of (f_n) to f on A, there exists an $N_1 \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| < \epsilon/3 \quad \forall\, x \in A, \quad \forall\, n \geq N_1.$$

Also, since $a_n \to L$, there exists an $N_2 \in \mathbf{N}$ such that

$$|a_n - L| < \epsilon/3 \quad \forall\, n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$ (or $N = N_1 + N_2$). For $n = N$, since $\lim_{x\to c} f_N(x) = a_N$, there exists $\delta > 0$ such that

$$|f_N(x) - a_N| < \epsilon/3 \quad \forall\, x \in A, \quad 0 < |x - c| < \delta.$$

Then, whenever $x \in A$ and $0 < |x - c| < \delta$, it follows that

$$|f(x) - L| \leq |f_N(x) - f(x)| + |f_N(x) - a_N| + |a_N - L| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Hence $\lim_{x\to c} f(x) = L$. □

Corollary 6.2 (Continuity of uniform limit function). *Let (f_n) and f be functions on A and let (f_n) converge uniformly to f on A. Assume for each $n \in \mathbf{N}$ the function f_n is continuous at a point $c \in A$. Then f is continuous at c as well.*

Proof. This follows from the previous theorem, but we give a direct proof.

Given $\epsilon > 0$, by the uniform convergence of (f_n) to f on A, there exists an $N \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| < \epsilon/3 \quad \forall\, x \in A, \quad \forall\, n \geq N.$$

For $n = N$, since f_N is continuous at $c \in A$, there exists a number $\delta > 0$ such that

$$|f_N(x) - f_N(c)| < \epsilon/3 \quad \forall\, x \in A, \quad |x - c| < \delta.$$

Then, whenever $x \in A$ and $|x - c| < \delta$, it follows that

$$\begin{aligned}|f(x) - f(c)| &\leq |f_N(x) - f(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

This proves the continuity of f at $c \in A$. □

Cauchy Criterion for Uniform Convergence. Like other Cauchy criteria, this criterion gives the necessary and sufficient condition for a sequence of functions to converge uniformly *without* knowing the limit function.

Theorem 6.3 (**Cauchy Criterion for Uniform Convergence**). *Let $f_n\colon A \to \mathbf{R}$ for each $n \in \mathbf{N}$. Then (f_n) converges uniformly on A if and only if, for each $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that*

$$|f_n(x) - f_m(x)| < \epsilon \quad \forall\, x \in A, \quad \forall\, n,\ m \geq N. \tag{6.3}$$

Proof. First assume (f_n) converges uniformly on A to a limit function f. Then, for each $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| < \epsilon/2 \quad \forall\, x \in A, \quad \forall\, n \geq N.$$

Hence, whenever $n,\ m \geq N$ and $x \in A$, it follows that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon,$$

proving (6.3).

We now assume condition (6.3) holds. Then, for each $x \in A$, the sequence $(f_n(x))$ is Cauchy; hence it converges to a limit $f(x) \in \mathbf{R}$. This defines a function $f\colon A \to \mathbf{R}$, which is the pointwise convergence limit of $f_n(x)$. The condition (6.3), used with ϵ replaced by $\epsilon/2$, implies that there exists an $N \in \mathbf{N}$ such that for all $x \in A$ and $n, m \geq N$ we have $|f_n(x) - f_m(x)| < \epsilon/2$. Hence

$$-\epsilon/2 < f_n(x) - f_m(x) < \epsilon/2 \quad \forall\, n,\ m \geq N.$$

We now fix $x \in A$ and $n \geq N$ and take the limit of sequence $(f_m(x))$ in this inequality. Use the order limit theorem, we have

$$-\epsilon/2 \leq f_n(x) - f(x) \leq \epsilon/2 \quad \forall\, n \geq N.$$

Hence we have proved that $|f_n(x) - f(x)| \leq \epsilon/2 < \epsilon$ holds for all $x \in A$ and $n \geq N$. This is nothing but the definition of the uniform convergence of (f_n) to f on A. □

6.2. Uniform Convergence and Differentiation

Theorem 6.4. *Let $f_n(x) \to f(x)$ pointwise on $[a, b]$ and assume each f_n is differentiable on an open interval containing $[a, b]$. If f_n' converges uniformly on $[a, b]$ to a function g, then f is differentiable and $f' = g$ on $[a, b]$.*

Proof. Let $c \in [a, b]$ be given. We want to show

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = g(c).$$

That is, given $\epsilon > 0$, we want to show that there exist a $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon \quad \text{for all } x \in [a, b] \text{ with } 0 < |x - c| < \delta.$$

Note that

$$\begin{aligned} \left| \frac{f(x) - f(c)}{x - c} - g(c) \right| &\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \\ &\quad + \left| \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) \right| + |f_n'(c) - g(c)|. \end{aligned} \tag{6.4}$$

The aim is to find a $\delta > 0$ such that each of the three terms on the righthand side of (6.4) is $< \epsilon/3$ for all $x \in [a,b]$ with $0 < |x-c| < \delta$.

The third term on the righthand side of (6.4) is independent of x, but depends on n. Since $f_n'(c) \to g(c)$, there exists an $N_1 \in \mathbf{N}$ such that

$$|f_n'(c) - g(c)| < \epsilon/3 \quad \forall\, n \geq N_1. \tag{6.5}$$

Let's handle the first term on the righthand side of (6.4) now. Note that by applying the MVT to function $f_m(x) - f_n(x)$ we have, for $x \neq c$,

$$\begin{aligned} \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} &= \frac{(f_m(x) - f_n(x)) - (f_m(c) - f_n(c))}{x-c} \\ &= f_m'(\alpha) - f_n'(\alpha) \end{aligned} \tag{6.6}$$

for some α between x and c (such a number α depends on many things: m, n, x, c). However, since (f_n') converges uniformly on $[a,b]$, by the Cauchy Criterion for Uniform Convergence, there exists an $N_2 \in \mathbf{N}$ such that

$$|f_n'(x) - f_m'(x)| < \epsilon/4 \quad \forall\, x \in [a,b], \quad \forall\, m, n \geq N_2.$$

We now use (6.6) to conclude that

$$\left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| = |f_m'(\alpha) - f_n'(\alpha)| < \epsilon/4$$

for all $x \neq c$ and all n, $m \geq N_2$. Take the limit as $m \to \infty$ and use the order limit theorem, and we have

$$\left| \frac{f(x) - f(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| \leq \epsilon/4 < \epsilon/3 \quad \forall\, x \neq c, \quad \forall\, n \geq N_2. \tag{6.7}$$

Now let $N = \max\{N_1, N_2\}$. Since f_N is differentiable at c with derivative $f_N'(c)$, there exists a $\delta > 0$ such that

$$\left| \frac{f_N(x) - f_N(c)}{x-c} - f_N'(c) \right| < \epsilon/3 \quad \forall\, x \in [a,b],\ |x-c| < \delta. \tag{6.8}$$

For this $\delta > 0$, whenever $x \in [a,b]$ and $0 < |x-c| < \delta$, with $n = N$ in (6.4) and using (6.7),(6.8) and (6.5), we have that

$$\begin{aligned} \left| \frac{f(x) - f(c)}{x-c} - g(c) \right| &\leq \left| \frac{f(x) - f(c)}{x-c} - \frac{f_N(x) - f_N(c)}{x-c} \right| \\ &\quad + \left| \frac{f_N(x) - f_N(c)}{x-c} - f_N'(c) \right| + |f_N'(c) - g(c)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

So $f'(c) = g(c)$. □

We also have a stronger result under a weaker assumption.

Theorem 6.5. *Let (f_n) be a sequence of differentiable functions defined on $[a,b]$, and assume (f_n') converges uniformly to a function g on $[a,b]$. If there exists a point $c \in [a,b]$ for which the sequence $(f_n(c))$ converges, then (f_n) must converge uniformly on $[a,b]$ to a function f that is differentiable on $[a,b]$ and satisfies $f'(x) = g(x)$ on $[a,b]$.*

Proof. If we have proved (f_n) converges uniformly to a function f on $[a,b]$, then the previous theorem will imply that f is differentiable on $[a,b]$ and $f'(x) = g(x)$ on $[a,b]$. The theorem is proved.

We now use the Cauchy criterion to show that (f_n) converges uniformly to a function f on $[a,b]$. Given any $\epsilon > 0$, by the uniform convergence of (f'_n), there exists an $N_1 \in \mathbf{N}$ such that

$$|f'_n(x) - f'_m(x)| < \epsilon/2(b-a) \quad \forall\, x \in [a,b], \quad \forall\, n,\, m \geq N_1.$$

Also, by the convergence of sequence $(f_n(c))$, there exists an $N_2 \in \mathbf{N}$ such that

$$|f_n(c) - f_m(c)| < \epsilon/2 \quad \forall\, n,\, m \geq N_2.$$

We use the MVT to function $f_n(x) - f_m(x)$ to obtain

$$(f_n(x) - f_m(x)) - (f_n(c) - f_m(c)) = (f'_n(\alpha) - f'_m(\alpha))(x-c)$$

for some α between x and c and hence $\alpha \in [a,b]$. Therefore, whenever $n,\, m \geq N_1$,

$$|(f_n(x) - f_m(x)) - (f_n(c) - f_m(c))| = |(f'_n(\alpha) - f'_m(\alpha))||x-c|$$

$$\leq |f'_n(\alpha) - f'_m(\alpha)|(b-a) < \frac{\epsilon}{2(b-a)}(b-a) = \epsilon/2.$$

Finally, let $N = \max\{N_1, N_2\}$. Then, whenever $n,\, m \geq N$ and $x \in [a,b]$, it follows that

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(c) - f_m(c))| + |(f_n(c) - f_m(c))|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence, by the Cauchy Criterion, (f_n) converges uniformly on $[a,b]$ to a function f. This proves the theorem. □

Example 6.5. Let $g_n(x) = \frac{x^n}{n}$ on $x \in [0,1]$. Again (g_n) uniformly converges to $g(x) = 0$, but $g'_n(x) = x^{n-1}$ only pointwise converges to $h(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$, not uniformly.

Example 6.6. Let $h_n(x) = (-1)^n \frac{\sin(nx)}{n}$ on $\mathbf{R}$. Then (h_n) converges uniformly to $h(x) = 0$ on $\mathbf{R}$. However $h'_n(x) = (-1)^n \cos(nx)$ does not converge for any $x \in \mathbf{R}$.

The previous examples indicate that for a sequence of differentiable functions (f_n) to converge uniformly to a differentiable function it is not necessary that (f'_n) converge uniformly (even pointwise).

6.3. Series of Functions

A series of functions is an infinite series of the form

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots,$$

where $f_1, f_2, \cdots$ are functions defined on a common set $A \subseteq \mathbf{R}$. Let the sequence of partial sums be defined by

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \quad n \in \mathbf{N}.$$

Definition 6.5. The series of functions $\sum_{n=1}^{\infty} f_n(x)$ is said to **converge pointwise on** A if the sequence $(s_n(x))$ converges pointwise on A; if $(s_n(x)) \to f(x)$ pointwise on A, then we write

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \quad x \in A.$$

If (s_n) converges uniformly on A then we say that the series $\sum_{n=1}^{\infty} f_n(x)$ **converges uniformly on** A.

Theorem 6.6. *Let f_n be continuous functions defined on a set $A \subseteq \mathbf{R}$, and assume $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A to a function f. Then f is continuous on A.*

Proof. Easy consequence of the case for sequence. □

Theorem 6.7 (**Term-by-Term Differentiation Theorem**)**.** *Let (f_n) be a sequence of differentiable functions defined on $[a, b]$, and assume $\sum_{n=1}^{\infty} f_n'$ converges uniformly to a function g on $[a, b]$. If there exists a point $c \in [a, b]$ for which the series $\sum_{n=1}^{\infty} f_n(c)$ converges, then the series $\sum_{n=1}^{\infty} f_n$ must converge uniformly to a differentiable function f on $[a, b]$ satisfying $f'(x) = g(x)$ on $[a, b]$.*

In other words, if $\sum_{n=1}^{\infty} f_n(c)$ converges at one point $c \in [a, b]$, and $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a, b]$, then the function

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is well-defined for all $x \in [a, b]$ and is differentiable on $[a, b]$ with

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \quad \forall\, x \in [a, b].$$

Proof. Use Theorem 6.5. □

Theorem 6.8 (**Cauchy Criterion for Uniform Convergence of Series**)**.** *A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on a set $A \subseteq \mathbf{R}$ if and only if for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that for all $n > m \geq N$,*

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \epsilon \quad \forall\, x \in A.$$

Proof. Use Theorem 6.3. □

Corollary 6.9 (**Weierstrass M-Test**)**.** *For each $n \in \mathbf{N}$, let f_n be a function defined on A satisfying $|f_n(x)| \leq M_n$ for all $x \in A$, where $M_n > 0$ is a real number. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.*

Proof. Use the previous Cauchy's Criterion. □

Example 6.7. Let

$$g(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{3^n} \quad (x \in \mathbf{R}).$$

First, since $|\frac{\cos(2^n x)}{3^n}| \leq \frac{1}{3^n}$ and $\sum \frac{1}{3^n}$ converges, by the M-test, the series defining $g(x)$ converges uniformly on $\mathbf{R}$. Since each function $\frac{\cos(2^n x)}{3^n}$ is continuous on $\mathbf{R}$, it follows that the function $g(x)$ is continuous on $\mathbf{R}$.

Note that the series of derivatives for $g(x)$ is

$$\sum_{n=1}^{\infty} -(\frac{2}{3})^n \sin(2^n x) \quad (x \in \mathbf{R}).$$

By the M-test, we also know that this series converges uniformly on $\mathbf{R}$. Hence, by the term-by-term differentiation theorem above, it follows that g is differentiable on $\mathbf{R}$, with

$$g'(x) = \sum_{n=1}^{\infty} -(\frac{2}{3})^n \sin(2^n x) \quad (x \in \mathbf{R})$$

being continuous on $\mathbf{R}$.

However, the series of derivatives for $g'(x)$ is given by

$$\sum_{n=1}^{\infty} -(\frac{4}{3})^n \cos(2^n x) \quad (x \in \mathbf{R}),$$

which does not converge. So we don't know whether g'' exists or not by the theorems we have learned.

***Nowhere differentiable continuous functions.**

Theorem 6.10. *There exists a continuous function on* $\mathbf{R}$ *which is not differentiable at every point.*

Proof. Define $\varphi(x) = |x|$ for $-1 \le x \le 1$ and extend φ to whole $\mathbf{R}$ as a periodic function of period 2; that is

$$\varphi(x+2) = \varphi(x) \quad \forall\, x \in \mathbf{R}.$$

Then, for all s and t in $\mathbf{R}$,

$$|\varphi(s) - \varphi(t)| \le |s-t|;$$

moreover, if no integer lies between s and t, then $|\varphi(s) - \varphi(t)| = |s-t|$. In particular, φ is continuous on $\mathbf{R}$. Define

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x) \quad \forall\, x \in \mathbf{R}.$$

Since $0 \le \varphi \le 1$, by the M-Test, this series converges uniformly to a continuous function, stilled called $f(x)$, on $\mathbf{R}$.

We now show that this function $f(x)$ is not differentiable at every $x \in \mathbf{R}$. So fix $x \in \mathbf{R}$. Let $m \in \mathbf{N}$ and

$$\delta_m = \pm \frac{1}{2} \cdot 4^{-m},$$

where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x+\delta_m)$; this is possible because if both intervals $(4^m(x-|\delta_m|), 4^m x)$ and $(4^m x, 4^m(x+|\delta_m|))$ contained integers then the length of interval $(4^m(x-|\delta_m|, 4^m(x+|\delta_m|))$, which is $2 \cdot 4^m |\delta_m| = 1$, would be greater than 1. Note that

$$\frac{f(x+\delta_m) - f(x)}{\delta_m} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \gamma_n,$$

where

$$\gamma_n = \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}.$$

If $n > m$, then $4^n\delta_m$ is an even integer, so that $\gamma_n = 0$. If $0 \le n \le m$, then $|\gamma_n| \le 4^n$; moreover, $|\gamma_m| = 4^m$ (this is due to the fact that no integer lies between $4^m x$ and $4^m(x+\delta_m)$). Therefore, we conclude that

$$\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| = \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n\right|$$

$$\ge \left(\frac{3}{4}\right)^m |\gamma_m| - \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n |\gamma_n| \ge 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m+1)$$

for all $m \in \mathbf{N}$. Hence $|\frac{f(x+\delta_m)-f(x)}{\delta_m}|$ is unbounded. Since $\delta_m \to 0$, it follows that the limit

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

does not exist, and thus f is not differentiable at x. □

6.4. Power Series

A **power series** is an infinite series of power functions:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots .$$

The point c is called the **center** of the power series and sequence (a_n) is called the **sequence of coefficients** of the power series.

Note that a power series always converges at its center c. In what follows, we will assume the center $c = 0$.

Theorem 6.11. *If a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \neq 0$, then it converges absolutely for all x satisfying $|x| < |x_0|$.*

Proof. If $\sum\limits_{n=0}^{\infty} a_n x_0^n$ converges, then the sequence $(a_n x_0^n)$ is bounded. Let $M > 0$ satisfy $|a_n x_0^n| \le M$ for all $n \in \mathbf{N}$. Assume $x \in \mathbf{R}$ satisfies $|x| < |x_0|$. Then

$$|a_n x^n| = |a_n x_0^n|\left|\frac{x}{x_0}\right|^n \le M\left|\frac{x}{x_0}\right|^n .$$

Since $|x/x_0| < 1$, by comparison with the geometric series, it follows easily that $\sum\limits_{n=0}^{\infty} |a_n x^n|$ converges; hence $\sum\limits_{n=0}^{\infty} a_n x^n$ converges absolutely. □

Radius of Convergence. Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series. Let

$$S = \left\{ x \in \mathbf{R} \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}.$$

There are three cases for this set S:

(a) $S = \{0\}$. In this case, the power series converges only at center 0.
(b) $S = \mathbf{R}$. In this case, the power series converges at every $x \in \mathbf{R}$.
(c) $S \neq \{0\}$ and $S \neq \mathbf{R}$.

In Case (c), there exist two points $x_0, x_1 \in \mathbf{R}$ such that $x_0 \neq 0$, $x_0 \in S$ and $x_1 \notin S$. By Theorem 6.11, for all $x \in \mathbf{R}$ with $|x| > |x_1|$, the power series diverges at x; otherwise, if it converges at some x with $|x| > |x_1|$ then by the theorem, $x_1 \in S$. Therefore, the set S is included in the interval $[-|x_1|, |x_1|]$ and hence S is a nonempty bounded set in $\mathbf{R}$. By the **AoC**, let

$$R = \sup S.$$

This R satisfies $|x_0| \leq R \leq |x_1|$; hence $R \in (0, \infty)$.

Lemma 6.12. *In Case (c), the power series converges at all x with $|x| < R$ and diverges at all x with $|x| > R$.*

Proof. 1. If $|x| < R = \sup S$, then there exists $x' \in S$ such that $x' > |x|$. Since the power series converges at x' and $|x| < |x'| = x'$, by Theorem 6.11, the power series converges at x.

2. If $|x| > R$, we show that the power series diverges at x. If not, assume the power series converges at x. Let $y = \frac{|x|+R}{2}$. Then $R < y < |x|$. Since the power series converges at x and $y < |x|$, by Theorem 6.11 again, the power series converges at y; hence $y \in S$. But $R = \sup S$; so $y \leq R$, a contradiction. □

Definition 6.6 (Radius of Convergence). Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with the set S defined as above. The **radius of convergence** of this power series is the number $R \in [0, \infty]$ defined as follows: $R = 0$ in Case (a); $R = \infty$ in Case (b); $R = \sup S \in (0, \infty)$ in Case (c).

Remark 6.7. Given a power series $\sum\limits_{n=0}^{\infty} a_n x^n$, by the **ratio test**, if the limit

$$\lim \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \left(\lim \left| \frac{a_{n+1}}{a_n} \right| \right) \quad \text{exists and is} < 1,$$

then the power series converges, and if $|x|(\lim |\frac{a_{n+1}}{a_n}|) > 1$. then the power series diverges.

So, suppose $\lim |\frac{a_{n+1}}{a_n}| = L$ exists. Then the power series $\sum a_n x^n$ converges if $|x| < 1/L$ and diverges if $|x| > 1/L$. Therefore the **radius of convergence** is $R = 1/L$. Clearly, this formula does not work when infinitely many terms $a_n = 0$.

In fact, by the **root test**, the **radius of convergence** is always given by

$$R = \frac{1}{\limsup\limits_{n \to \infty} \sqrt[n]{|a_n|}},$$

which involves the **upper limit** of sequences. However, this formula will not be needed for our lecture and the homework.

Example 6.8. For the power series $\sum\limits_{n=0}^{\infty} n!\, x^n$, the nth-term is $a_n = n! x^n$. If $x \neq 0$, then

$$\left| \frac{a_{n+1}}{a_n} \right| = (n+1)|x| \to \infty,$$

and hence $|a_n| \to \infty$. Therefore, $\sum a_n$ diverges. So the given power series only converges at $x = 0$ and diverges for all $x \neq 0$. The radius of convergence for this power series is $R = 0$.

Example 6.9. For power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

the nth-term is $a_n = \frac{x^n}{n!}$. Hence

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$$

for all $x \neq 0$; thus, the power series converges absolutely for all $x \neq 0$, and certainly it always converges for $x = 0$. Therefore the given series converges absolutely for all x. The radius of convergence for this power series is $R = \infty$.

Example 6.10. In Case (c) above, the set S must be only one of the four intervals

$$[-R, R], \;\; [-R, R), \;\; (-R, R], \;\; (-R, R),$$

and each of the four cases can happen.

(a) The power series $\sum_{n=1}^{\infty} x^n/n^2$ has the radius of convergence $R = 1$ and converges exactly on $[-1, 1]$.

(b) The power series $\sum\limits_{n=1}^{\infty} x^n/n$ has the radius of convergence $R = 1$ and converges exactly on $[-1, 1)$.

(c) The power series $\sum\limits_{n=1}^{\infty} (-x)^n/n$ has the radius of convergence $R = 1$ and converges exactly on $(-1, 1]$.

(d) The power series $\sum\limits_{n=0}^{\infty} x^n$ has the radius of convergence $R = 1$ and converges exactly on $(-1, 1)$.

Uniform Convergence of Power Series and Abel's Theorem.

Theorem 6.13. *If a power series* $\sum\limits_{n=0}^{\infty} a_n x^n$ *converges* absolutely *at a point* $x_0 \neq 0$, *then it converges uniformly on the interval* $[-|x_0|, |x_0|]$.

Proof. If $|x| \leq |x_0|$ then $|a_n x^n| \leq |a_n x_0^n| := M_n$. Then use Weierstrass's M-Test. □

Lemma 6.14 (Abel's Lemma). *Let* (b_n) *satisfy* $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$ *and let* (a_n) *satisfy*

$$|a_1 + a_2 + \cdots + a_n| \leq A, \quad \forall\, n \in \mathbf{N},$$

for a constant $A > 0$. *Then, for all* $n \in \mathbf{N}$,

$$|a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n| \leq A b_1.$$

Proof. Let $s_0 = 0$ and $s_k = a_1 + a_2 + \cdots + a_k$ for $k \in \mathbf{N}$. Then $a_k = s_k - s_{k-1}$ for all $k \in \mathbf{N}$. Hence

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (s_k - s_{k-1}) b_k = \sum_{k=1}^{n} s_k b_k - \sum_{k=1}^{n} s_{k-1} b_k$$

$$= \sum_{k=1}^{n} s_k b_k - \sum_{k=0}^{n-1} s_k b_{k+1} = \sum_{k=1}^{n-1} s_k b_k + s_n b_n - \sum_{k=1}^{n-1} s_k b_{k+1} = \sum_{k=1}^{n-1} s_k (b_k - b_{k+1}) + s_n b_n.$$

Since $|s_k| \leq A$ and $b_k - b_{k+1} \geq 0$, it follows that

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \left|\sum_{k=1}^{n-1} s_k (b_k - b_{k+1}) + s_n b_n\right|$$

$$\le \sum_{k=1}^{n-1} |s_k|(b_k - b_{k+1}) + |s_n| b_n \le A \sum_{k=1}^{n-1}(b_k - b_{k+1}) + Ab_n = Ab_1.$$

□

Theorem 6.15 (**Abel's Theorem**). *If a power series* $\sum\limits_{n=0}^{\infty} a_n x^n$ *converges at* $x = R > 0$*, then it converges uniformly on the interval* $[0, R]$.

If a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ *converges at* $x = -R < 0$*, then it converges uniformly on the interval* $[-R, 0]$.

Proof. Assume $\sum\limits_{n=0}^{\infty} a_n R^n$ converges and we use the Cauchy criterion to show $\sum\limits_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, R]$; the case at $x = -R$ is completely similar.

Given $\epsilon > 0$, from the convergence of $\sum\limits_{n=0}^{\infty} a_n R^n$, there exists a number $N \in \mathbf{N}$ such that

$$|a_{m+1}R^{m+1} + a_{m+2}R^{m+2} + \cdots + a_n R^n| < \epsilon/2, \quad \forall\, n > m \ge N.$$

Let $x \in [0, R]$. For any fixed $m \ge N$, consider sequence $B_n = (\frac{x}{R})^{m+n}$ and sequence $A_n = a_{m+n}R^{m+n}$ with $n \in \mathbf{N}$. Then the sequences (B_n) and (A_n) satisfy the conditions in Abel's Lemma above with $A = \epsilon/2$. Hence, for all $n > m$,

$$\left| a_{m+1}R^{m+1}(\frac{x}{R})^{m+1} + a_{m+2}R^{m+2}(\frac{x}{R})^{m+2} + \cdots + a_n R^n (\frac{x}{R})^n \right| \le (\frac{x}{R})^{m+1}\epsilon/2 < \epsilon.$$

Therefore,

$$|a_{m+1}x^{m+1} + a_{m+2}x^{m+2} + \cdots + a_n x^n| < \epsilon, \quad \forall\, x \in [0, R], \quad n > m \ge N.$$

This proves the uniform convergence of $\sum\limits_{n=0}^{\infty} a_n x^n$ on $[0, R]$. □

The Success of Power Series. Given a power series $\sum\limits_{n=0}^{\infty} a_n x^n$, the **differentiated series** $\sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ is also a power series.

Theorem 6.16. *If a power series* $\sum\limits_{n=0}^{\infty} a_n x^n$ *converges for all* $x \in (-R, R)$*, then the differentiated series* $\sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ *converges for all* $x \in (-R, R)$ *as well.*

Proof. Let t be such that $|x| < t < R$. The conclusion follows from the identity

$$|n a_n x^{n-1}| = \frac{1}{t}\left(n \left|\frac{x^{n-1}}{t^{n-1}}\right|\right) |a_n t^n|.$$

Since $r = |x|/t < 1$, it follows that $(nr^{n-1}) \to 0$. This can be shown by the convergence of series $\sum nr^n$ using the ratio-test. □

Theorem 6.17. *If a power series converges pointwise on a set* A*, then it converges uniformly on any compact subset of* A.

Proof. When A contains one of R or $-R$, where R is the radius of convergence, we need the Abel's theorem. For all other cases, the theorem can be proved without it. □

Theorem 6.18. *Assume the power series*

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges on a set A. Then, g is continuous on A and differentiable on any open interval $(-R, R) \subseteq A$ with the derivative given by the term-by-term differentiation

$$g'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad x \in (-R, R).$$

Moreover, g is infinitely differentiable on $(-R, R)$, and the higher-order derivatives can be obtained via the term-by-term differentiation of the previous differentiated power series.

Proof. Only for the continuity of g at possibly the end-points of the interval of convergence is the previous theorem needed. All other conclusions can be proved without using the Abel's theorem. $\square$

Example 6.11. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (x \in \mathbf{R}).$$

Then, from the term-by-term differentiation,

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = f(x).$$

Also $f(0) = 1$. Let $g(x) = f(x)e^{-x}$. Then $g(0) = 1$ and $g'(x) = f'(x)e^{-x} - f(x)e^{-x} = 0$ for all $x \in \mathbf{R}$. So $g(x) \equiv 1$; that is, $f(x) = e^x$ for all $x \in \mathbf{R}$. In particular, with $x = 1$,

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e = 2.718281828459045\ldots.$$

Example 6.12. Let

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

The series has radius of convergence $R = 1$ and converges **exactly** for $x \in (-1, 1]$. We can prove this by just using the fact $g(-1)$ diverges and $g(1)$ converges. (Explain why.) Hence g is continuous on $(-1, 1]$ and is differentiable on $(-1, 1)$. The derivative is given by

$$g'(x) = \sum_{n=1}^{\infty} (-x)^{n-1}, \quad -1 < x < 1.$$

So, by the geometric series, $g'(x) = \frac{1}{1+x}$ for $-1 < x < 1$. But $g(0) = 0$. Hence

$$\begin{aligned} g(x) &= \int_0^x g'(t)\,dt = \int_0^x \frac{1}{1+t}\,dt \\ &= [\ln|1+t|]_0^x = \ln|1+x| = \ln(1+x); \end{aligned}$$

that is,

$$g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x), \quad -1 < x < 1.$$

This identity also holds when $x = 1$ since g is continuous at $x = 1$. So we have that the value of the **alternative harmonic series** is given by

$$g(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

6.5. Taylor Series

The power series defines an infinitely differentiable function on its open interval of convergence. Given an infinitely differentiable function on an open interval, can we express the function as a power series centered at an interior point of the interval? Such a series is called the Taylor series of the function centered at the given point.

Let f be an infinitely differentiable function defined on $(-R, R)$. Suppose f equals a power series centered at 0 as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots, \quad x \in (-R, R).$$

Then $a_0 = f(0)$. By the term-by-term differentiation, we have, for all $k = 1, 2, 3, \cdots$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n x^{n-k}, \quad x \in (-R, R).$$

Hence $f^{(k)}(0) = k! a_k$. Therefore, with $0! = 1$, we have

$$a_k = \frac{f^{(k)}(0)}{k!}, \quad k = 0, 1, 2, \cdots.$$

This shows that if a function f equals a power series near 0 then the coefficients a_k of the power series must be given by the above formula.

Definition 6.8. Let f be an infinitely differentiable function near a point $a \in \mathbf{R}$. Then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad \text{near } a \tag{6.9}$$

is called the **Taylor series** of f about a. When $a = 0$, the Taylor series is also called the **Maclaurin series**; but we will still call it the Taylor series.

Lagrange's Remainder Theorem. Does the Taylor series of f converge to f near a? We assume $a = 0$. The following result is useful to answer such a question.

Theorem 6.19 (Lagrange's Remainder Theorem)**.** *Let f have all derivatives up to order $N+1$ on $(-R, R)$ and define*

$$S_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x^k \quad \forall x \in (-R, R).$$

Then, given any $0 < |x| < R$, there exists a number c with $|c| < |x|$ such that the **error term**

$$E_N(x) = f(x) - S_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}.$$

Therefore,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N)}(0)}{N!}x^N + \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}. \tag{6.10}$$

This formula is also called **Taylor's Formula** *for f at 0 with degree $N+1$.*

Proof. Without loss of generality, assume $0 < x < R$. Consider function $E_N(t) = f(t) - S_N(t)$. Then

$$E_N^{(n)}(0) = 0 \quad \text{for all } n = 0, 1, 2, \cdots, N,$$

and $E_N^{(N+1)}(t) = f^{(N+1)}(t)$ for all $t \in (-R, R)$. So, by the **Generalized Mean-Value Theorem** repeatedly, we have $x > x_1 > x_2 > \cdots > x_{N+1} > 0$ such that

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N'(x_1)}{(N+1)x_1^N} = \frac{E_N''(x_2)}{(N+1)Nx_2^{N-1}} = \cdots = \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)N\cdots 2\cdot 1} = \frac{f^{(N+1)}(c)}{(N+1)!},$$

and this proves the theorem with $c = x_{N+1}$. □

EXAMPLE 6.13. Show

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \quad \forall\, x \in \mathbf{R}.$$

Proof. Let $f(x) = \sin x$. Then

$$f^{(2n)}(x) = (-1)^n \sin x, \quad f^{(2n+1)}(x) = (-1)^n \cos x$$

for all $n = 0, 1, 2, \ldots$. Therefore the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

is exactly the Taylor series of $\sin x$ about 0. Let $S_N(x)$ be the partial sum up to power x^N. Then

$$S_{2k}(x) = S_{2k+1}(x) = \sum_{n=0}^{k} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

By Lagrange's Remainder Theorem,

$$|\sin x - S_{2k+1}(x)| = |E_{2k+1}(x)| = \left|\frac{f^{(2k+2)}(c)}{(2k+2)!}x^{2k+2}\right| \le \frac{1}{(2k+2)!}|x|^{2k+2} \to 0$$

as $k \to \infty$ for all $x \in \mathbf{R}$. This proves the convergence; moreover it gives the **error estimate**

$$\left|\sum_{n=0}^{k} \frac{(-1)^n}{(2n+1)!}x^{2n+1} - \sin x\right| \le \frac{|x|^{2k+2}}{(2k+2)!}$$

for all $x \in \mathbf{R}$ and all $k = 0, 1, \cdots$. □

A Counterexample. The Taylor series of f may not be equal to f near a.

EXAMPLE 6.14. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Then f is infinitely differentiable at all $x \neq 0$ with $f'(x) = e^{-1/x^2}(2x^{-3})$, $f''(x) = e^{-1/x^2}(-6x^{-4} + 4x^{-6})$ and in general

$$f^{(n)}(x) = e^{-1/x^2} \sum_{k=0}^{3n} a_k^n x^{-k} \quad (n = 0, 1, \cdots), \;\; x \neq 0,$$

where a_k^n's are constants (some are zero).

It is a good exercise to show that f has all orders of derivatives at 0 and $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \cdots$.

(**Hint:** Use $e^y \geq y^m/m!$ for all $y > 0$ and $m \in \mathbf{N}$ to prove $\lim_{x \to 0} x^{-k} e^{-1/x^2} = 0$ for each integer $k \geq 0$.)

Therefore, the Taylor series of f about 0 is identically zero; obviously $f(x) \neq 0$ whenever $x \neq 0$. This shows that f is not equal to its Taylor series near 0.